

Adaptation of the Landau-Migdal Quasiparticle Pattern to Strongly Correlated Fermi Systems

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A quasiparticle pattern advanced in Landau's first article on Fermi liquid theory is adapted to elucidate the properties of a class of strongly correlated Fermi systems characterized by a Lifshitz phase diagram featuring a quantum critical point (QCP) where the density of states diverges. The necessary condition for stability of the Landau Fermi Liquid state is shown to break down in such systems, triggering a cascade of topological phase transitions that lead, without symmetry violation, to states with multi-connected Fermi surfaces. The end point of this evolution is found to be an exceptional state whose spectrum of single-particle excitations exhibits a completely flat portion at zero temperature. Analysis of the evolution of the temperature dependence of the single-particle spectrum yields results that provide a natural explanation of classical behavior of this class of Fermi systems in the QCP region.

I. PREAMBLE

Once upon a time at a traditional meeting of nestlings of the Migdal school, A. B. began his speech with words that stunned the audience.

Until recent times I was proud that I have never published an incorrect article. But is it actually a true cause for pride? Can an experienced mountaineer take pride in not having broken any ribs, or a professional motorcyclist, that his legs are still intact? The correct answer is no! It just means that these people may have run to the best of their abilities, but certainly not more.

This memorial article, dedicated to a great physicist and a great man, is devoted to a problem first discussed around 20 years ago.¹⁻³ The cited works considered the possibility of a breakdown of the conventional Landau-Migdal quasiparticle pattern⁴⁻⁷ of phenomena observed in Fermi liquids (FL), associated specifically with rearrangement of the $T = 0$ Landau quasiparticle momentum distribution $n_F(p) = \theta(p_F - p)$.

Over the last decade, experimental studies of non-Fermi liquid (NFL) behavior of strongly correlated systems have extended the frontiers of low-temperature condensed matter physics.⁸⁻¹⁵ During the same period, a number of theorists have engaged in efforts to extend the frontiers of FL theory with the aim of explaining this anomalous behavior.¹⁶⁻²⁸

There is a famous Migdal Correspondence Criterion for judging new theories, which boils down to this:

First an foremost, the proposed theory must be able to match results of previous, well-tested theoretical descriptions. It is only of secondary importance that it matches the relevant experimental data. Why so? Because as a rule, the experiments of burning interest were performed yesterday, and therefore the results obtained may be flawed, whereas theoretical physics stands as body of knowledge created and honed by a multitude over three

hundred years.

The extended quasiparticle picture to be reviewed and discussed here meets the Migdal Criterion. It reduces naturally to the standard FL picture when dealing with conventional Fermi liquids. At the same time, convincing experimental evidence supporting this extension of the Landau-Migdal vision, while present, remains scarce—possibly because of the short time the relevant experimental programs have been in operation.^{9,14,29,30} Even so, one cause for the lack of unambiguous correlations between available experimental data and corresponding theoretical results may be conceptual (or technical) errors made by the theorists in developing the new theory, errors that are yet to be exposed. In this case, we should not be too upset, since according to Migdal's provocative challenge to his assembled group, such errors tell us that the struggling theorists, being professionals, did, after all, run beyond the breaking point of their abilities.

II. ROUTES TO BREAKDOWN OF STANDARD FERM-LIQUID THEORY

Any theory has own limits of applicability, and FL theory is no exception. Conventionally these limits are imputed to violation of Pomeranchuk stability conditions.³¹ Such violation is associated with second-order phase transitions that are accompanied with jumps of the specific heat $C(T)$ and cusps of the spin susceptibility $\chi(T)$. Upon approach to the transition point, spontaneous creation and enhancement of fluctuations suppresses the value of the z factor that determines the quasiparticle weight in the single-particle state at the Fermi surface. At the transition point, z vanishes, signaling a breakdown of the FL quasiparticle picture.^{32,33}

A different domain where predictions of FL theory prove to be fallacious has been discovered and explored

during the last decade. This is the regime of the so-called quantum critical point (QCP) revealed in experimental studies of 2D liquid ^3He and heavy-fermion metals. In this domain, the specific heat $C(T)$ and spin susceptibility $\chi(T)$, both proportional to the density of states $N(T)$ (and hence to the effective mass M^*), are found to diverge as the temperature T goes to zero.^{8–15}

The presence of a QCP is the hallmark that distinguishes strongly correlated Fermi systems from those with weak or moderate correlations. In this respect, three-dimensional (3D) liquid ^3He , for which M^* remains finite at any density, belongs to the class of systems with moderate correlations. Its 2D counterpart, on the other hand, is assigned to the class of strongly correlated Fermi systems based on evidence for the existence of a QCP provided by experimental studies of dense ^3He films.^{8–12}

It is well to emphasize that the divergence of the specific heat $C(T)$ that is observed in the QCP region has little in common with the jumps in $C(T)$ inherent in second-order phase transitions. Notwithstanding this fact, theoretical explanations of the failure of FL theory in the QCP region have commonly linked its apparent breakdown with attendant second-order phase transitions.^{13,32–35} However, as they occur in the QCP region, such transitions are found to be quite atypical. In most cases, even the structure of the order parameters remains unknown, and the properties of states beyond the QCP defy explanation within the standard scaling theory of second-order phase transitions. One is led to conclude that the viability of such transitions as triggers of the observed NFL behavior is problematic.

In this article, we pursue another explanation of NFL behavior in the QCP region, attributing the breakdown of standard FL theory to violation of the *necessary stability condition* for the Landau state.¹⁶ The salient feature of the proposed scenario is rearrangement of single-particle degrees of freedom driven by *topological* phase transitions (TPT's). This mechanism stands in direct contrast to the violation of *sufficient* Pomeranchuk stability conditions,³¹ which entail rearrangement of *collective degrees of freedom* in second-order phase transitions.

Since the topological scenario does not implicate the violation of any Pomeranchuk stability condition, the catastrophic suppression of the quasiparticle weight posited in the collective scenario for the QCP does not take place, and failure of standard FL theory in the QCP region must have another explanation. Indeed, the fundamental FL formula

$$\epsilon(p) = v_F(p - p_F) \equiv p_F(p - p_F)/M^* \quad (1)$$

for the single-particle spectrum $\epsilon(p)$ measured from the Fermi surface, becomes powerless when the effective mass M^* diverges. If such a divergence is present, additional terms of the Taylor expansion of $\epsilon(p)$ must be added to the right side of Eq. (1) (see Ref. 22), giving rise to a change of sign of the Fermi velocity. This in its turn signals a violation of the necessary stability condition for the conventional Landau state (as seen in Sec. V.B), resulting

in a *topological* rearrangement of the Fermi surface that necessitates modification of the standard FL formalism.

III. REPRISE OF THE STANDARD FL QUASIPARTICLE PICTURE

To set the stage for discussion of the required modifications, we recall that the heart of the Landau-Migdal quasiparticle picture is the postulate that there exists a one-to-one correspondence between the totality of real, decaying single-particle excitations of the actual Fermi liquid and a system of immortal interacting quasiparticles. There are two facets of this correspondence. First, the number of quasiparticles is equal to the given number N of particles, a condition expressed as

$$\text{Tr} \int n(\mathbf{p}) dv = \frac{N}{V} \equiv \rho, \quad (2)$$

where $n(\mathbf{p})$ is the quasiparticle momentum distribution, ρ is the density, Tr implies summation over spin and isospin variables, and $dv = d\mathbf{p}/(2\pi)^D$ is a volume element in a momentum space of dimension D . Second, all thermodynamic quantities, notably the ground state energy E and entropy S , are treated as functionals of the quasiparticle momentum distribution $n(\mathbf{p})$. In particular, the entropy S of the real system is given by the ideal-Fermi-gas combinatorial expression

$$\frac{S}{V} = -\text{Tr} \int [n(\mathbf{p}) \ln n(\mathbf{p}) + (1 - n(\mathbf{p})) \ln(1 - n(\mathbf{p}))] dv. \quad (3)$$

In homogeneous matter, a standard variational procedure based on Eq. (3) and involving restrictions that impose conservation of the particle number and energy, leads to the result⁴

$$n(p) = \left[1 + e^{\epsilon(p)/T} \right]^{-1}. \quad (4)$$

While this relation resembles the corresponding Fermi-Dirac formula for the ideal Fermi gas, the quasiparticle energy $\epsilon(p)$ measured from the chemical potential does not coincide with the bare single-particle energy $\epsilon_p^0 = p^2/(2M) - \mu$. Instead, it is given by the variational derivative

$$\epsilon(p) = \delta\Omega/\delta n(p) \quad (5)$$

of the thermodynamic potential $\Omega = E - \mu N$.

It is significant that the most seminal article cited by Landau in his pioneering works^{4,5} was authored by A. B. Migdal,³⁶ who had advanced an idea that allows unambiguous determination of the Fermi momentum p_F of the quasiparticle system. In his numerous papers and books, A. B. also adapted the Landau quasiparticle concept to finite systems where the phenomenological spectrum (1) becomes more complicated due to its dependence on the shape of the self-consistent field acting on a quasiparticle

in an inhomogeneous medium. Indeed, his famous book⁶ completed the creation of the FL formulation through the exposition of a quantitative theory of the behavior of Fermi liquids in external fields. This book has become a touchstone for modern theoretical description of the properties of atomic nuclei. Moreover, together with A. I. Larkin he developed a quantitative theory of superfluid Fermi liquids.³⁷ It is in recognition of these splendid achievements that the FL quasiparticle theory is often referred to as the Landau-Migdal quasiparticle pattern.

In principle, the quasiparticle spectrum $\epsilon(p)$ can be evaluated by means of the relation^{4,5,38,39}

$$v(\mathbf{p}) = \frac{\partial \epsilon(p)}{\partial \mathbf{p}} = \frac{\mathbf{p}}{M} + \int f(\mathbf{p}, \mathbf{p}_1) \frac{\partial n(p_1)}{\partial \mathbf{p}_1} dv_1, \quad (6)$$

stemming from the assertion that the single-particle energy $\epsilon(p)$ is a functional of the quasiparticle momentum distribution $n(p)$. Alternatively, this equation may be derived from the Galilean invariance of the system Hamiltonian and gauge invariance.^{40,41} From this coincidence we infer that the spectrum $\epsilon(p)$ is, indeed, a functional of n , thereby reinforcing the validity of the basic Landau postulate that the ground state energy E and other thermodynamic quantities can be treated as functionals of n .

Eq.(6) provides a nonlinear integral equation for self-consistent determination of the quasiparticle spectrum $\epsilon(p)$ and the quasiparticle momentum distribution (4), treating the Landau interaction function $f(\mathbf{p}, \mathbf{p}_1)$ as phenomenological input. It is worth noting that the correct relation between the function $f(\mathbf{p}, \mathbf{p}_1)$ and the scattering amplitude Γ was first established by Migdal.⁴² This function turns out to be the so-called ω -limit of $\Gamma(\mathbf{p}, \mathbf{p}_1, \mathbf{k}, \omega)$, where the energy transfer ω and momentum transfer k both tend to zero, but in such a way that $k/\omega \rightarrow 0$. In this limit, only regular Feynman diagrams contribute to f in full force; consequently the interaction function f cannot be evaluated within the FL formalism itself. Accordingly, f is commonly specified by a set of phenomenological parameters, namely the harmonics of its Legendre polynomial expansion.

Following Landau (cf. Eq. (4) in Ref. 4), one postulates that at $T = 0$, solutions of Eq. (6) arrange themselves in such a way that the Fermi velocity v_F *always* keeps a positive value. Necessarily, then, the quasiparticle momentum distribution $n(p, T = 0)$ coincides with the Fermi step $n_F(p) = \theta(p_F - p)$ appropriate to a noninteracting Fermi gas. In turn, the relation (2) reduces to the famous Landau-Luttinger theorem, expressed simply as

$$\rho = \frac{p_F^3}{3\pi^2} \quad (7)$$

for the case of 3D homogeneous matter. The resulting quasiparticle picture, in which a system of fermions is treated as a “gas of interacting quasiparticles”⁶ works flawlessly for conventional Fermi liquids, including 3D

liquid ³He, the matter inside atomic nuclei, and the electron liquid in alkali metals, where the correlations are moderate or weak. However, as already indicated, this picture begins to fail upon entry into the QCP region, where the effective mass M^* diverges.

IV. THEORETICAL ALTERNATIVES FOR DIVERGENCE OF THE EFFECTIVE MASS

Discrepancies between FL predictions and experimental data were first observed in films of liquid ³He at relatively high areal density. In stark contrast to the ordinary FL behavior exhibited by the bulk liquid, the 2D data for the spin susceptibility $\chi(T)$ and Sommerfeld ratio $\gamma = C(T)/T$ of specific heat $C(T)$ to temperature T soar upward as $T \rightarrow 0$,^{8–12} whereas both quantities must become constant in FL theory. Analogous behavior has been discovered and documented in strongly correlated electron systems of solids^{13–15} and in the 2D electron gas as realized in MOSFETs.^{43–47}

Such a behavior is ascribed to divergence of the effective mass M^* , first revealed theoretically in microscopic calculations of the single-particle spectrum $\epsilon(p)$ of the dilute 3D homogeneous electron gas^{48,49} (for the 2D case, see Ref. 50). Two different sources for divergence of M^* in nonsuperfluid Fermi systems can be identified in terms of the textbook formula

$$\frac{M}{M^*} = z \left[1 + \left(\frac{\partial \Sigma(p, \epsilon)}{\partial \epsilon_p^0} \right)_0 \right], \quad (8)$$

where the quasiparticle weight z in the single-particle state is given by $z = [1 - (\partial \Sigma(p, \epsilon)/\partial \epsilon)_0]^{-1}$, with Σ representing the mass operator. Here and henceforth, the subscript 0 indicates that a function of p and ϵ is evaluated at the Fermi surface.

As seen from this formula, one way in which a divergence of M^* could arise (first discussed by Doniach and Engelsberg³²) is through divergence of the derivative $(\partial \Sigma(p, \epsilon)/\partial \epsilon)_0$ at a critical density ρ_c where some second-order phase transition occurs. Another option invokes a *change of sign* of the sum $1 + (\partial \Sigma(p, \epsilon)/\partial \epsilon_p^0)_0$ at a critical density ρ_∞ . Correspondingly, there are two different scenarios for the QCP. In the first scenario—the most popular until recently—it is the energy dependence of the self-energy $\Sigma(p, \epsilon)$ that plays the decisive role. In this *collective scenario*, the posited divergence of $(\partial \Sigma(p, \epsilon)/\partial \epsilon)_0$ at the points of second-order phase transitions results in a vanishing renormalization factor z , and the QCP is identified with the end point of the line $T_N(\rho)$ traced in the phase diagram by a second-order phase transition. The essence of this scenario is captured in the maxim¹³ that in the vicinity of any second-order phase transition, “Quasiparticles get heavy and die.”

In the alternative *topological* scenario, it is instead the momentum dependence of the mass operator $\Sigma(p, \epsilon)$ that gives rise to the QCP at a critical density ρ_∞ where the

sum $1 + (\partial\Sigma(p, \varepsilon)/\partial\varepsilon_p^0)_0$ changes sign. Let us recall that on the disordered side of the QCP, system properties *obey standard FL theory*. This fact, as we will see, is incompatible with the collective scenario. Indeed, divergence of the effective mass imposes some restriction on the first harmonic f_1 entering Eq. (6). By way of illustration, we address a familiar 3D case in which simple manipulations performed on Eq. (6) at the critical QCP density $\rho_\infty = p_\infty^3/3\pi^2$ yield

$$\frac{v_F(\rho_\infty)}{v_F^0} \equiv \frac{M}{M^*(\rho_\infty)} = 1 - \frac{1}{3}F_1^0(\rho_\infty) = 0, \quad (9)$$

where $v_F^0 = p_F/M$.

In principle, given the bare interaction potential V , the harmonic f_1 can be evaluated within a microscopic approach to FL theory (for examples, see Refs. 16,51), in which this quantity is a continuous functions of the density ρ . Then the left-hand side of Eq. (9) must change sign beyond the point where the relation (9) is met, implying that the Fermi velocity v_F , evaluated with the Landau quasiparticle momentum distribution $n_F(p)$, *necessarily* changes sign at the QCP. This behavior conflicts with the collective scenario for the QCP, in which such a sign change is impossible. We conclude that if FL theory is applicable on the disordered side of the QCP, the collective scenario is not relevant to the QCP. The sign change results in the violation of the necessary stability condition for the Landau state. Nevertheless, as we will see in Sec.V.B, the original Landau quasiparticle picture holds on both sides of the QCP, implying that in contrast to the collective scenario, the QCP itself and the line $T_N(\rho)$ of putatively associated second-order phase transitions are *separated* from each other.

In some influential theoretical articles devoted to the physics of the QCP, it has been claimed that switching on the interactions between particles fails to produce a significant momentum dependence in the effective interaction function f , and hence that the topological mechanism is irrelevant. This assertion cannot withstand scrutiny. The natural measure of the strength of momentum-dependent forces in the medium is provided by the dimensionless first harmonic of the interaction function $f(\mathbf{p}, \mathbf{p}_1)$ of Landau theory. To avoid any misunderstandings it is worth replacing the divergent effective mass M^* appearing in the dimensionless Landau harmonic $F_1 = f_1 p_F M^*/\pi^2$ by the free-particle mass to obtain $F_1^0 = f_1 p_F M/\pi^2$ as a relevant characteristics of the strength of the momentum dependence of the effective interactions. In a system such as 3D liquid ^3He where the correlations are of moderate strength, the result $F_1^0 \geq 2.0$ for this measure extracted from specific-heat data is already rather large. The data on 2D liquid ^3He are yet more damaging to the claim of minimal momentum dependence, since the effective mass is found to *diverge* in dense films,⁸⁻¹² implying that $F_1^0 > 3.0$. In considering the occurrence of QCP phenomena in strongly correlated systems of *ionic* crystals, it should be borne in mind that the electron effective mass is greatly enhanced

due to electron-phonon interactions that subserve polaron effects.⁵²⁻⁵⁴

The irrelevance of the collective scenario may be drawn with respect to the notion that a Pomeranchuk instability drives the QCP. To be definite, the argument may be framed in terms of a ferromagnetic phase transition, for which the associated Pomeranchuk stability condition reads

$$1 + G_0(\rho_\infty) \equiv 1 + g_0(\rho_\infty) \frac{p_\infty M^*(\rho_\infty)}{\pi^2} = 0. \quad (10)$$

Here we set $\rho = \rho_\infty$ because in the collective scenario, the point of divergence of the effective mass coincides with the point of a second-order phase transition.

The crucial point is that at the QCP, both the relations (9) and (10) must be met *simultaneously*. Thus if at the QCP, the Pomeranchuk stability condition is violated then the right-hand sides of both Eq. (10) and Eq. (9) must vanish at the same density ρ_∞ . However, this vanishing can only happen *accidentally* or with the aid of an external magnetic field. Otherwise, we encounter a dilemma: either $F_1^0(\rho_\infty) = 3$ and hence the effective mass diverges while the spin susceptibility remains finite; or else $1 + G_0(\rho_\infty) = 0$, and the spin susceptibility diverges while the effective mass remains finite. Anyway, in both the cases, the end point of the line $T_N(\rho)$ is separated from the QCP. Such a separation of phases was first observed in 2D liquid ^3He ¹² and was also recently documented in precision experiments on heavy-fermion metals.^{55,56}

In studies of the class of heavy-fermion metals, the information most valuable for an understanding of their properties is drawn from studies of those compounds in which localized magnetic moments of ions forming the crystal lattice interact with narrow electron bands located near the Fermi level. The full theory of this so-called Kondo lattice problem is far from complete (for example, see the monograph of Hewson.⁵⁷) During the last decade certain results from this theory were applied in the analysis of the quantum critical phenomena in heavy-fermion metals.^{15,58-61} In the corresponding scenario for the QCP, called the Kondo breakdown model, the Kondo resonance is destroyed at the QCP, causing the Kondo temperature T_K to vanish and the FL picture to be recovered on the disordered side of the QCP. Quantitatively, this model, with the dynamical exponent $\mathcal{Z} = 3$,⁶¹ differs from the standard spin-density wave (SDW) model where $\mathcal{Z} = 2$, providing more realistic critical indexes. Notwithstanding this fact, the Kondo breakdown scenario for the QCP is burdened by the same shortcomings as the SDW scenario. In all, the weight of the above considerations compels us to “place the cart and the horse” in developing a viable theory of the QCP, that is, to shift the focus from collective mechanisms to topological rearrangement of the Fermi surface as the origin of NFL behavior and the key player in the QCP regime.

V. EMERGENCE OF FLAT BANDS AS A UNIVERSAL FEATURE OF STRONGLY CORRELATED FERMİ SYSTEMS

In standard FL theory, applicable to weakly and moderately correlated Fermi systems, the $T = 0$ ground-state momentum quasiparticle distribution $n(p)$ is the Fermi step $n_F(p) = \theta(p_F - p)$. In this case, solution of the Landau relation (9) for the quasiparticle spectrum $\epsilon(p)$ is obviated, being reduced to a simple momentum integration. The solutions for $\epsilon(p)$ prove to be continuous functions, with all derivatives existing everywhere in momentum space. On the other hand, in strongly correlated Fermi systems exhibiting a QCP, the basic FL formula (1) becomes fallacious, together with the asserted property $n(p) = n_F(p)$. As we shall see, the *necessary* condition for stability of the Landau state is *inevitably* violated at some point in the QCP domain, implying that the minimum of the functional $E[n]$ is displaced from the Fermi step $n_F(p)$ to another point in the admissible space of momentum distributions specified by the Pauli restriction $0 \leq n(p) \leq 1$.

A. Case study: Rearrangement of the Landau state in a simple model

In practice, numerical solution of the basic relation (6) should demonstrate whether the true $T = 0$ ground-state momentum distribution $n_*(p)$ coincides with the Fermi step $n_F(p)$ or with something more interesting. However, we need not resort to cumbersome numerical calculations to elucidate the rearrangement of the Landau state in strongly correlated Fermi systems. To illustrate this point, consider as an example a simple model where the energy functional $E(n)$ has the form

$$E(n) = \text{Tr} \int \frac{p^2}{2M} n(p) dv + \frac{1}{2} \text{Tr}_1 \text{Tr}_2 \int n(p_1) n(p_2) f(\mathbf{p}_1 - \mathbf{p}_2) dv_1 dv_2 \quad (11)$$

that results in the following relation

$$\epsilon(p) = \frac{p^2}{2M} - \mu + \int f(\mathbf{p} - \mathbf{p}_1) n(p_1) dv_1 \quad (12)$$

between the quasiparticle momentum distribution $n(p)$ and the single-particle spectrum $\epsilon(p)$. In what follows we focus on key factors that promote a rearrangement of the Landau state as the correlations gain strength, assuming the interaction function to be as follows: $f(q) = \lambda/q$, with the zero and first harmonics coinciding with each other: $f_0 = f_1 = \lambda/p_F$. It is this form of f that was employed in the first article¹ where the phenomenon of flattening of single-particle spectra was uncovered. According to Ref. 1, the necessary stability condition for the Landau state is violated at the critical value

$$F_1^0(\rho_\infty) = \frac{f_1 p_F M}{\pi^2} \equiv \frac{M \lambda_{\text{QCP}}}{\pi^2} = 3. \quad (13)$$

Comparison of this result with Eq. (9) shows that the effective mass M^* does diverge at this point.

A straightforward estimate for the FL ground-state energy per particle is obtained by inserting the Fermi step n_F into Eq. (11) to yield

$$\frac{E(n_F)}{N} = \frac{3p_F^2}{10M} \left(1 + \frac{2F_1^0}{3} \right), \quad (14)$$

If F_1^0 is large, the interaction term is dominant. To diminish this term and thereby lower $E(n)$, the momentum distribution must be spread as much as possible, until this is rendered counterproductive due to the consequent increase of the kinetic term.

To estimate the effect of a rearrangement from $n(p)$ to $n_*(p)$, let us assume the new ground-state momentum distribution takes the form $n_*(p) = \nu \theta(p_f - p)$, with $\nu < 1$. Particle-number conservation then implies $p_f = \nu^{-1/3} p_F$, and the kinetic energy is increased by a factor $(p_f/p_F)^2$. If the interaction function f were momentum-independent, no rearrangement would occur. In the model posed, however, $f(q)$ falls off as $1/q$, and therefore the replacement $n_F(p) \rightarrow n_*(p)$ leads to the following result

$$\frac{E(n_*)}{N} = \frac{3p_F^2}{10M} \left[\frac{p_f^2}{p_F^2} + \frac{2p_F}{3p_f} F_1^0 \right]. \quad (15)$$

This energy estimate is minimized with respect to the parameter p_f by the ratio

$$\frac{p_f}{p_F} = \left(\frac{F_1^0}{3} \right)^{1/3}, \quad (16)$$

thus fixing ν . This ratio must exceed unity, lest the Pauli restriction be violated; hence we surmise that the results obtained are meaningful provided $F_1^0 > 3$. Referring to Eq. (9), we then infer that if this inequality is met, the Fermi velocity v_F does change its sign, and the Landau state is indeed destabilized. With these results, the energy advantage of the rearranged state, $\Delta E_* = E(n_*) - E(n_F)$, is readily estimated:

$$\Delta E_* = -\frac{3p_F^2}{10M} \left[\left(\frac{F_1^0}{3} \right)^{1/3} - 1 \right]^2 \left[1 + 2 \left(\frac{F_1^0}{3} \right)^{1/3} \right]. \quad (17)$$

Since the variational procedure is performed on a wider class of solutions, the rearrangement ensures a *negative* supplement ΔE_* .

Naturally, one cannot clarify all details of the rearrangement of the Landau state by means of simple variational estimates. It will be seen below that in correlated Fermi systems where the interaction function is strongly repulsive, the momentum distribution n_* that minimizes the functional $E(n)$ does in fact dive inside the definitional domain D of $n(p)$, rather than switching between one boundary point of this domain and the other. In this

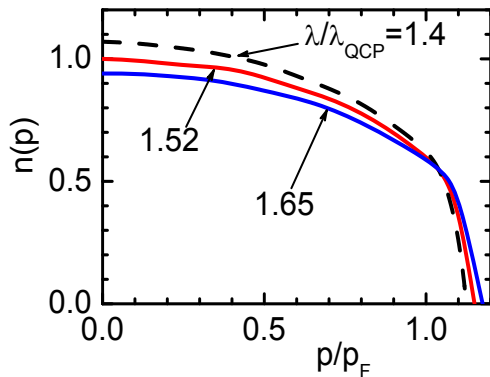


FIG. 1: Quasiparticle filling $n(p)$ calculated for $\kappa = 0.07 p_F$ and different values of coupling constant λ in the model with interaction $f(q) = \lambda(q^2 + \kappa^2)^{-1}$.

case, the true ground state momentum distribution $n_*(p)$ is found as a solution of the variational equation¹

$$\frac{\delta E(n(p))}{\delta n(p)} - \mu = 0. \quad (18)$$

As a concrete illustration, Fig. 1 presents results from numerical calculations performed in Ref. 19 for the quasiparticle filling determined from Eq. (18), employing $f(q) = \lambda(q^2 + \kappa^2)^{-1}$ with $\kappa = 0.07 p_F$. The resulting quasiparticle momentum distribution $n_*(p)$, smooth over the whole region $0 < p < p_f$, is found by means of a regularization procedure that is effective in treating ill-posed problems. As applied here, this procedure involves the introduction of a very small additional local interaction between quasiparticles of the form $U_0 \delta(\mathbf{p} - \mathbf{p}_1)$. Eq. (18) then has the specific expression

$$\frac{p^2}{2M} + U_0 n_*(p) + \lambda \int \frac{n_*(p_1)}{(\mathbf{p} - \mathbf{p}_1)^2 + \kappa^2} dv_1 - \mu = 0, \quad (19)$$

and was solved by a conventional grid method, taking $U_0/\varepsilon_F^0 = 10^{-3}$. As seen in Fig. 1, a smooth solution of Eq. (19) exists for every λ value. However, *this solution does not satisfy the Pauli principle* until λ attains a critical value $\lambda_{cr} = 1.52 \lambda_{QCP}$, where λ_{QCP} corresponds to the first topological transition associated with the QCP. At $\lambda > \lambda_{cr}$, the Pauli principle becomes immaterial. In the illustrative case considered, the Fermi volume is found to expand or swell by as much as 30%.

B. Flattening of the spectrum of single-particle excitations in strongly correlated Fermi systems

One striking feature of the phenomenon of rearrangement of the Landau state at $\lambda > \lambda_{cr}$ is a profound flattening of the spectrum of single-particle excitations $\epsilon(p)$. The emergence of the flat band can be illuminated without elaborate calculations. The flattening of $\epsilon(p)$ stems

from the necessary condition for stability of the ground state,¹⁶ which requires that the change of the ground-state energy E_0 produced by any admissible variation of the momentum distribution $n(p)$ be nonnegative

$$\delta E_0 = \int \epsilon(p; n_F) \delta n(p) dv \geq 0. \quad (20)$$

Starting from the Landau state, the admissible variations of $n(p)$ from $n_F(p)$ must obey the Pauli principle, being negative at $p < p_F$ and positive at $p > p_F$. In the standard FL theory, the signs of $\epsilon(p)$ and $p - p_F$ always coincide and therefore the Landau state is stable. However, when we address the rearranged filling of momentum states represented by $n_*(p)$, the quasiparticle occupancies differ from the FL values 0 and 1 in an interval $[p_i, p_f]$; accordingly the admissible variations of $n_*(p)$ in this range do not have a definite sign. The only way to avoid violation of the inequality (20) in this situation is to make the spectrum $\epsilon(p)$ *completely flat* in the domain $p_i < p < p_f$. This phenomenon, originally called fermion condensation¹ and later swelling the Fermi surface,³ or emergence of flat bands, was discovered 20 years ago. It was recently rediscovered by Lee⁶² in a different context, investigating the finite-charge-density sector of conformal field theory (CFT) based on the AdS/CFT gravity/gauge duality.

C. Breakdown of the analyticity of solutions of the Landau equation for the spectrum $\epsilon(p)$

From the mathematical standpoint, the FL functional $E_{FL}(n)$ is equivalent to any functional of the Thomas-Fermi (TF) family of functionals that depend on the density $\rho(\mathbf{r})$, even though $E_{FL}(n)$ and $E_{TF}(\rho)$ refer to different definitional domains of the arguments n and ρ . For the ensuing development, it is convenient to write $E_{TF}(\rho)$ in the generalized form

$$E_{TF}(\rho) = \tau(\rho(r)) + \int \rho(r) U(r) d\mathbf{r} + \frac{1}{2} \iint \rho(\mathbf{r}_1) \rho(\mathbf{r}_2) V(\mathbf{r}_1 - \mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2, \quad (21)$$

with the normalization condition

$$\int \rho(r) d\mathbf{r} = Z, \quad (22)$$

where the *finite* number Z is the atomic charge. In Eq. (21), we identify $\tau(\rho)$ as the electron kinetic energy, $U(r)$ as an external field, and $V(r)$ as the two-particle interaction, vanishing for $r \rightarrow \infty$. The minimum of the functional (21) is determined from the standard variational condition

$$\epsilon(r) \equiv \frac{\delta E_{TF}(\rho)}{\delta \rho(r)} - \mu = 0. \quad (23)$$

The solution $\rho(r)$ of this equation must be everywhere nonnegative; in addition, it must vanish at $r \rightarrow \infty$ to meet the normalization condition (22). Explicitly, one finds

$$\frac{p_F^2(r)}{2M} - Z\frac{e^2}{r} + e^2 \int \frac{\rho(r_1)}{|\mathbf{r} - \mathbf{r}_1|} d\mathbf{r}_1 - \mu = 0, \quad (24)$$

where $p_F(r) = (3\pi^2\rho(r))^{1/3}$.

The density $\rho(r)$ is known to decay rapidly as r increases beyond the TF radius $\simeq 1/(Me^2)$, implying that the integral on the left side of Eq. (24) and the external field cancel each other at distances greatly exceeding this radius. Since μ vanishes in a neutral atom, the asymptotic conditions are met, and there exists a solution $\rho(r)$ that is everywhere continuous, with all its derivatives defined.⁶³

The preceding argument breaks down for the case of an ionized atom. This is evident already from the singly-ionized case: all terms on the left in Eq. (24) vanish as $r \rightarrow \infty$ *but one*, namely the chemical potential μ , which has a finite value.⁶³ Thus, Eq. (24) cannot be satisfied at $r \rightarrow \infty$. We conclude that this equation only applies within a *finite domain* defined by $r < R$, with R treated as the boundary of the ionized atom beyond which $\rho(r)$ vanishes identically.⁶³ The equations determining the minimum of the functional (21) now become

$$\begin{aligned} \frac{p_F^2(r)}{2M} - Z\frac{e^2}{r} + e^2 \int_0^R \frac{\rho(r_1)}{|\mathbf{r} - \mathbf{r}_1|} d\mathbf{r}_1 - \mu &= 0, \quad r \leq R, \\ \rho(r) &\equiv 0, \quad r > R. \end{aligned} \quad (25)$$

Having obtained the corresponding density distribution, the function $\epsilon(r)$ of Eq. (23) is evaluated in closed form as

$$\begin{aligned} \epsilon(r) &= 0, \quad r \leq R, \\ \epsilon(r) &= -Z\frac{e^2}{r} + e^2 \int_0^R \frac{\rho(r_1)}{|\mathbf{r} - \mathbf{r}_1|} d\mathbf{r}_1 - \mu, \quad r > R. \end{aligned} \quad (26)$$

The energy $\epsilon(r)$ vanishes identically only at $r \leq R$, whereas at $r > R$ it changes somewhat with the observation point r . Since any analytic function that vanishes identically in some domain must be identically zero everywhere, we see that analytic solutions of the TF variational problem *simply do not exist* in the case of an ionized atom.

In the preceding development, we have demonstrated that the Thomas-Fermi problem is ill posed, by witnessing the breakdown of analyticity due to the impossibility of reconciling (i) the variational equation that is to determine the solution by functional minimization with (ii) the prescribed asymptotic behavior of the desired solution. Similarly, the task of finding a satisfactory minimum of the Landau functional $E_{\text{FL}}(n)$ also belongs to the class of ill-posed problems. Since the first term in

Eq. (19) diverges at $p \rightarrow \infty$, there are no solutions of the Landau variational problem that are continuous with derivatives everywhere in momentum space. Therefore the domain of solution must be restricted, as in the TF problem (though in momentum space rather than coordinate space) leading to the following equation analogous to first of Eqs. (25)

$$\frac{p^2}{2M} + \int f(\mathbf{p} - \mathbf{p}_1) n_*(p_1) dv_1 - \mu = 0, \quad p \in \mathcal{C}. \quad (27)$$

A NFL solution $n_*(p)$ continuous in the momentum domain \mathcal{C} shows *no Migdal jump* at the Fermi momentum $p = p_F$. Correspondingly, in this domain the basic property $n^2(p) = n(p)$ obeyed by the standard $T = 0$ FL quasiparticle momentum distribution *no longer holds*. Outside the domain \mathcal{C} , the spectrum $\epsilon(p)$ ceases to be flat, and accordingly the distribution $n_*(p)$ coincides with the Fermi step, being 1 for occupied states and 0, for empty ones. As stated above, the phenomenon of “swelling” of the Fermi surface implies that it expands from a line to a surface in 2D, and from a surface to a volume in 3D. In other words, beyond the point of fermion condensation, the single-particle spectrum $\epsilon(p)$ acquires a completely flat portion $\epsilon(p) \equiv 0$ in the momentum interval $p_i < p < p_f$ (the so-called *fermion condensate* (FC)).

With the aid of the variational condition (18), one can understand that the system with the FC *cannot* be treated as a gas of interacting quasiparticles, because in case a particle with momentum $\mathbf{p} \in \mathcal{C}$ is added to the system, the addition results in a rearrangement of the *whole* distribution function $n_*(p)$ in the \mathcal{C} region. This is an essential aspect of the modification of the FL quasiparticle picture required to describe strongly correlated Fermi systems.

D. Systems hosting a FC as a special class of Fermi liquids

The flattening of the spectrum $\epsilon(p)$ in systems with a FC is reflected in the structure of the single-particle Green function $G(p, \varepsilon)$ on the imaginary axes in the limit $\varepsilon \rightarrow 0$, where²

$$G(p, \varepsilon) = \frac{1}{\varepsilon}, \quad p \in \mathcal{C}. \quad (28)$$

This makes the essential difference in evaluation of the *topological charge*, given in the 2D case by the integral^{2,64}

$$\mathcal{N} = \int_{\gamma} (G(p, \varepsilon))^{-1} \partial_l G(p, \varepsilon) \frac{dl}{2\pi i}, \quad p \in \mathcal{C}, \quad (29)$$

where the Green function is considered on the imaginary energy axis $\varepsilon = i\xi$ and the integration is performed over a *small* contour γ in the (\mathbf{p}, ξ) -space embracing the Fermi line. For conventional Fermi liquids and states with a multi-connected Fermi surface, the topological charge \mathcal{N}

is an integer, whereas for a states having a completely flat portion in the spectrum $\epsilon(p)$, integration over the contour embracing the Fermi line yields a *half-odd-integral* value of \mathcal{N} .² Accordingly, from the topological point of view, systems hosting a FC form a *special class* of Fermi liquids.

In dealing with the whole Fermi-surface region, Eq. (28) needs a slight generalization. To explain this modification, we introduce a minute temperature T . It is then natural to write the FC Green function in the form

$$G(p, \epsilon, T) = \frac{1 - n_*(p, T)}{\epsilon + i\gamma} + \frac{n_*(p, T)}{\epsilon - i\gamma}, \quad p \in \mathcal{C}, \quad (30)$$

familiar in low-temperature physics, the occupation numbers $n_*(p, T)$ being given by the Landau formula (4). In writing this formula the ratio γ/ϵ is assumed to be being much less than unity, which applies at least for systems having a small proportion of FC.

E. Low-temperature expansion in systems with a FC

It is a key feature of the phenomenon of fermion condensation that a tiny elevation of temperature has practically no effect on the $T = 0$ momentum distribution, so that Eq. (4) can be inverted to obtain³

$$\epsilon(p, T \rightarrow 0) = T \ln \frac{1 - n_*(p)}{n_*(p)}, \quad p \in \mathcal{C}. \quad (31)$$

This means that at extremely low temperatures, the dispersion of the single-particle spectrum $\epsilon(p)$ in the FC domain is proportional to T , thus allowing one to identify (or redefine) the FC as the totality of single-particle states having a linear-in- T dispersion of their associated spectrum $\epsilon(p, T)$. As a matter of fact, the formula (31) is instrumental to developing low-temperature expansions of the basic quantities $\epsilon(p, T)$ and $n(p, T)$. To clarify how such expansions are generated, we first decompose the quasiparticle momentum distribution $n(p, T)$ as follows:

$$n(p, T) = n_*(p) + T\nu(p, T), \quad (32)$$

where $n_*(p)$ is the $T = 0$ solution of Eq. (12). To facilitate the analysis we assume that the FC occupies that part of momentum space lying between $p = 0$ to $p = p_f$, so that $n_*(p)$ vanishes identically at $p \geq p_f$.

Upon inserting the decomposition (32) into Eq. (12) and replacing there $\epsilon(p)$ by $T \ln[(1 - n(p))/n(p)]$ we arrive, as before, at the system of two equations

$$\begin{aligned} \ln \frac{1 - n(p, T)}{n(p, T)} &= \int f(p, p_1) \nu(p_1, T) dv_1, \quad p \leq p_f, \\ -\ln \nu(p, T) &= z_*(p) + \int f(p, p_1) \nu(p_1, T) dv_1, \quad p \geq p_f, \end{aligned} \quad (33)$$

where $z_*(p) = \epsilon_*(p)/T$, with

$$\epsilon_*(p) = \frac{p^2}{2M} + \int f(p, p_1) n_*(p_1) dv_1 - \mu, \quad p \geq p_f. \quad (34)$$

In deriving the second of Eqs. (33), the logarithmic term has been simplified by the replacement $1 - \nu(p) \rightarrow 1$, noting that the quantity $\nu(p)$ is small everywhere in momentum space. The behavior of $\nu(p)$ is readily explicated in two regions of momentum space on either side of the boundary point p_f . In the FC region, upon neglecting ν on the left-hand side of the first of Eqs. (33), this equation becomes linear, and upon introducing the low- T expansion, we have

$$\nu(p, T) = \alpha_0 + \alpha_1 T + \dots, \quad p \in \mathcal{C}. \quad (35)$$

The coefficients $\alpha_0 \sim O(1), \alpha_1, \dots$ may be evaluated in closed form. This expansion begins to fail near the boundary point p_f , where the ratio $f n_*(p)/T$ becomes less than unity and the logarithmic structure of the left-hand members of Eq. (33) exhibits its influence in the emergence of an additional factor $\ln(1/T)$ in $\nu(T)$. Outside the FC region, the solution

$$\nu(p, T) \simeq e^{-\epsilon_*(p)/T} \quad (36)$$

is easily found, provided $\epsilon_*(p)/T \gg 1$. Near the boundary point p_f where $\epsilon_*(p)/T \simeq 1$, the simple formula (36) fails, and the quantity $\nu(T)$ again experiences a moderate enhancement, which can be evaluated only numerically.

The impact of such enhancement on the thermodynamic quantities of a system hosting a FC is not significant, since the range of the domain where the logarithmic enhancement of $\nu(T)$ takes place is too small. As an example, let us evaluate the specific heat. We recall that at the QCP, the Sommerfeld ratio

$$\gamma(T) = \int \frac{\epsilon}{T} \frac{\partial n(\epsilon)}{\partial T} \frac{dp}{d\epsilon} d\epsilon \quad (37)$$

diverges at $T \rightarrow 0$ due to the divergence of the density of states, which is proportional to the effective mass $M^*(T)$. In systems with a FC, this divergence is seen to disappear because the major term $n_*(p)$ of the sum (32), being T -independent, does not contribute to $\gamma(T)$. With this result, upon inserting Eq. (35) into Eq. (37), the contribution of the FC region to $\gamma(T)$ is evaluated in a straightforward matter, retaining the first term $\alpha_0 \simeq 1$ of the expansion (35). Since, as always in thermodynamic calculations, the condition $|\epsilon|/T \simeq 1$ applies, the result so obtained turns out to be T -independent just as in standard FL theory—a characteristic feature of systems with a FC, first disclosed in Ref. 3.

Let us now estimate the contribution to the Sommerfeld ratio γ from an integration region in which the function ν is logarithmically enhanced. Since in this region, the limits of integration are restricted by the same condition $|\epsilon| \simeq |\epsilon_*| \leq T$, we surmise that the magnitude of this contribution is determined by the product of the

factor T and the average value of the factor $dp/d\epsilon_*(p)$. Analysis shows that the latter is indeed enhanced in the boundary region; one has $dp/d\epsilon_*(p) \propto T^{-\alpha}$, but with $\alpha < 1$. Collecting all factors, we find that at $T \rightarrow 0$, the contribution to γ of the regions adjacent to the boundary point p_f is suppressed as $T^{1-\alpha} \ln(1/T)$. Accordingly, we conclude that the emergence of a FC in significant proportion produces a great suppression of the QCP behavior of the Sommerfeld ratio.

F. Fate of the Landau-Luttinger theorem in systems with a FC

Here we demonstrate that the Landau-Luttinger theorem (7) breaks down in systems hosting a FC. It will be instructive to pinpoint the flaw that emerges in pursuit of the standard Luttinger argument.⁶⁵ The analysis begins with expression of the particle number N as the integral³⁹

$$\frac{N}{V} = -2 \int_{-\infty}^0 \int \frac{\partial}{\partial \varepsilon} \ln \left[\frac{G(p, \varepsilon)}{G^*(p, \varepsilon)} \right] \frac{d\mathbf{p} d\varepsilon}{(2\pi)^4 i}. \quad (38)$$

In conventional Fermi liquids, this integral is proportional to the difference $\ln(G(p, \varepsilon=0)/G^*(p, \varepsilon=0)) - \ln(G(p, \varepsilon=-\infty)/G^*(p, \varepsilon=-\infty))$. In turn, the value of the ratio $\ln(G(p, \varepsilon)/G^*(p, \varepsilon))$ is proportional to the argument $\varphi(\varepsilon)$ of the complex function $\ln G(p, \varepsilon)$. One has³⁹ $\varphi(-\infty) = \pi$ independently of the presence or absence of a FC in the system. Since $\text{Im} G^{-1}(p, \varepsilon) = 0$, a nonzero value of the integral in momentum space is obtained by integrating over a domain where $\text{Re} G(p, \varepsilon=0) > 0$, yielding the customary result $p_F^3/3\pi^2$. However, in the FC region, the function $\text{Re} G^{-1}(p, \varepsilon=0)$ vanishes identically, and for states belonging to the FC, the result, contained in the expression

$$\frac{N}{V} = \frac{p_i^3}{3\pi^2} - 2 \int_{p_i}^{p_f} \int_{-\infty}^0 \frac{\partial}{\partial \varepsilon} \ln \left[\frac{G(p, \varepsilon)}{G^*(p, \varepsilon)} \right] \frac{d\mathbf{p} d\varepsilon}{(2\pi)^4 i}, \quad (39)$$

becomes uncertain. It is treated by dividing the region of energy integration into two intervals, namely from $-\infty$ to a small negative value δ , and from δ to 0. According to Eq. (30), $G(p, \varepsilon=\delta)$ is simply $1/\epsilon$, in agreement with the property (28). Then the integral over the first of the two intervals vanishes, since, at $\varepsilon = \delta$, the argument of the ratio $\ln(G(p, \varepsilon=\delta)/G^*(p, \varepsilon=\delta))$ coincides with that at $\varepsilon = -\infty$. As a result, we are left with the integral over the second interval, containing $\ln(G(p, \varepsilon=0)/G^*(p, \varepsilon=0))$, which can be evaluated by going around the pole of the Green function (30). As a result, we are led to the Landau formula (2), which now reads explicitly

$$\frac{N}{V} = \frac{p_i^3}{3\pi^2} + 2 \int_{p_i}^{p_f} n_*(p) \frac{d\mathbf{p}}{(2\pi)^3}. \quad (40)$$

Thus we conclude that in systems having a FC, the Landau-Luttinger theorem (7) is violated, yet the Landau postulate (2) remains intact.

VI. MERGING OF SINGLE-PARTICLE LEVELS IN FINITE FERMI SYSTEMS

A phenomenon analogous to swelling of the Fermi surface in infinite Fermi systems also exists in finite systems. The systems implicated include atomic nuclei, whose quantitative description within the FL framework was elaborated by A. B. Migdal.⁶ In their conventional wisdom, textbooks teach us that under variation of input parameters, two neighboring single-particle levels may repel or cross one other. However, this familiar dichotomy overlooks a third alternative: levels can in fact merge.⁶⁶

This phenomenon, missing in the theory of finite Fermi systems, is made possible by the variation of single-particle energies with level occupation numbers—a property central to Landau theory. The primary condition for merging to occur is that the Landau-Migdal interaction function f is repulsive in coordinate space, which holds for the effective nn and pp interactions in the nuclear interior⁶ and for the electron-electron interaction in atoms.

A. Schematic model of merging of single-particle levels

Following Ref. 66, consider a schematic model involving three equidistant neutron levels, separated by an energy distance D in an open shell of a spherical nucleus. The levels are denoted $-$, 0 , and $+$, in order of increasing energy. The single-particle energies ϵ_λ and wave functions $\varphi_\lambda(\mathbf{r}) = R_{nl}(r)\Phi_{jlm}(\mathbf{n})$ are solutions of

$$[p^2/2M + \Sigma(\mathbf{r}, \mathbf{p})]\varphi_\lambda(\mathbf{r}) = \epsilon_\lambda \varphi_\lambda(\mathbf{r}), \quad (41)$$

where Σ denotes the self-energy. In even-even spherical nuclei, which in their ground states have total angular momentum $J = 0$ due to pairing correlations, the energies ϵ_λ are independent of the magnetic quantum number m associated with the total single-particle angular momentum j . We suppose that the level $-$ is filled, the level $+$ is empty, and N neutrons are added to the level 0 , changing the density $\rho(r)$ by $\delta\rho(r) = NR_{n_0 l_0}^2(r)/4\pi$.

In what follows we shall retain only a major, spin- and momentum-independent part V of the self-energy Σ and a dominant, $\delta(r)$ -like portion of the Landau-Migdal interaction function f . Accordingly, the FL relation between Σ and ρ responsible for the variation of $\epsilon_\lambda(n)$ with n reduces to⁶

$$\delta V(r) = f[\rho(r)]\delta\rho(r). \quad (42)$$

When particles are added to the system, all energy levels are shifted somewhat, but the level that receives

the particles is affected more strongly than the others. For the sake of simplicity, the diagonal and nondiagonal matrix elements of f are assigned the respective values

$$\begin{aligned} u &= \int R_{nl}^2(r) f[\rho(r)] R_{nl}^2(r) r^2 dr / 4\pi, \\ w &= \int R_{nl}^2(r) f[\rho(r)] R_{n_1 l_1}^2(r) r^2 dr / 4\pi, \end{aligned} \quad (43)$$

independently of the quantum numbers $nl, n_1 l_1$.

Based on these assumptions and results, the dimensionless shifts $\xi_k(N) = [\epsilon_k(N) - \epsilon_k(0)] / D$ for $k = 0, +, -$ are given by

$$\xi_0(N) = n_0 U, \quad \xi_+(N) = \xi_-(N) = n_0 W, \quad (44)$$

where $n_k = N_k / (2j_k + 1)$ is the occupation number of the level k , $U = u(2j_0 + 1)/D$, and $W = w(2j_0 + 1)/D$. It is readily verified that if $f p_F M / \pi^2 \sim 1$, where $p_F = \sqrt{2M\epsilon_F}$ and ϵ_F is the Fermi energy, then the first of the integrals (43) has a value $u \simeq \epsilon_F / A$ and therefore $U \sim 1$, since $D \sim \epsilon_F / A^{2/3}$ in spherical nuclei.

According to Eqs. (44) at $(U - W) > 1$, the difference $d(N) = 1 + \xi_+(N) - \xi_0(N)$ changes sign at $n_{0c} = 1/(U - W)$, before filling of the level $+$ is complete. At $n_0 > n_{0c}$, in the standard scenario provided by Hartree-Fock theory, all added quasiparticles must resettle into the empty single-particle level $+$. However, not all of the quasiparticles can take part in the migration process, since the situation would then be reversed, and the roles of the levels interchanged: the formerly empty level, lying above the formerly occupied one, would have the maximum positive energy shift, rendering migration impossible. Thus, the standard Fermi-liquid filling scenario, which prescribes that *one and only one* single-particle level lying exactly at the Fermi surface can remain unfilled, while all others must be completely occupied or empty, encounters a contradiction.

This contradiction is resolved as follows.⁶⁶ Migration occurs until the single-particle energies of the two levels in play coincide. As a result, both of the levels, 0 and $+$, become *partially* occupied—an impossible situation for the standard Landau state. Solution of the problem reduces to finding the minimum of the relevant energy functional

$$E_0 = \epsilon_0(0)N_0 + \epsilon_+(0)N_+ + \frac{1}{2} [u(N_0^2 + N_+^2) + 2wN_0N_+] \quad (45)$$

with $N_k = \sum_m n_{km}$, through a variational condition

$$\frac{\delta E_0}{\delta n_{0m}} = \frac{\delta E}{\delta n_{+m_1}} = \mu, \quad \forall m, m_1, \quad (46)$$

where μ is the chemical potential. Eqs. (46) are conveniently rewritten as conditions

$$\begin{aligned} \epsilon_0(N) &= \epsilon_0(0) + N_0 u + N_+ w = \mu, \\ \epsilon_+(N) &= \epsilon_+(0) + N_0 w + N_+ u = \mu \end{aligned} \quad (47)$$

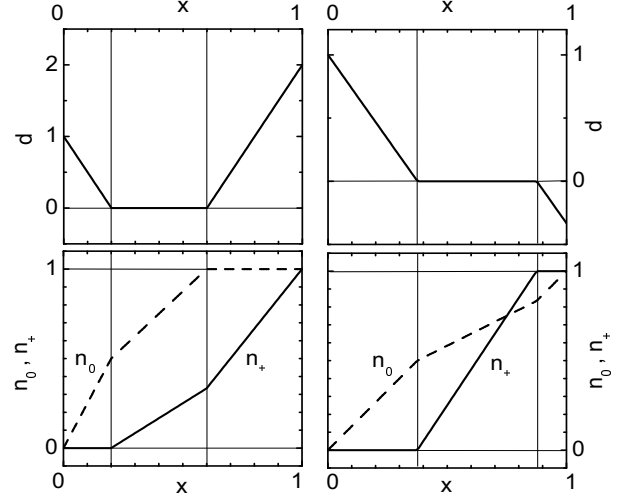


FIG. 2: Top panels: Dimensionless distance $d = (\epsilon_+ - \epsilon_0)/D$ between levels $+$ and 0 as a function and 0 as a function of the ratio $x = N/(2j_0 + 2j_+ + 2)$. Lower panels: Occupation numbers n_k for levels 0 and $+$. Input parameters: $U = V = 3$, $W = 1$. For the left column, the ratio $r \equiv (2j_0 + 1)/(2j_+ + 1) = 2/3$; for the right, $r = 3$.

for coincidence of the single-particle energies ϵ_0 and ϵ_+ , which, at $N > N_c = (2j_0 + 1)/(U - W)$, yield $N_0 = \frac{1}{2}(N + N_c)$ and $N_+ = \frac{1}{2}(N - N_c)$.

Results from numerical calculations are plotted in Fig. 2, which consists of two columns, each made up of three plots. The upper panels show the dimensionless ratio $d(x) = [\epsilon_+(x) - \epsilon_0(x)] / D$ versus $x = N/(2j_0 + 2j_+ + 2) \in [0, 1]$. The lower panels give the occupation numbers $n_+(x)$ and $n_0(x)$. We observe that there are three different regimes: in two of them $d \neq 0$ and there exist well-defined single-particle excitations, and in the third, the energies of the levels 0 and $+$ coincide at zero. Passage through the three regimes can be regarded as a second-order phase transition, with the occupation number n_+ treated as an order parameter.

The single-particle levels remain merged until one of them is completely filled. If the level 0 fills first, as in the left column of Fig. 2, then under further increase of N , quasiparticles fill the level $+$, signaling that the distance $d(N)$ again becomes positive. This behavior resembles the repulsion of two levels of the *same symmetry* in quantum mechanics, although here one deals with the single-particle levels of *different symmetry*. In a case where level $+$ becomes fully occupied before the level 0 does, as in the right column, the distance $d(N)$ becomes negative, and the two levels just cross each other at this point. Interestingly, analogous behavior has been discovered recently⁶⁷ in numerical calculations of single-particle spectra of multilayered systems.

B. Competition between shell-model effects and merging of single-particle levels

As is well known that the stability of atomic nuclei hinges critically upon negative shell corrections to the Weizsäcker formula for ground-state energies. These corrections are maximal in magic nuclei such as Sn¹³² and Pb²⁰⁸, where the density of single-particle levels near the Fermi surface drops sharply due to the existence of an energy gap between the last occupied and first unoccupied levels. Such behavior is in complete agreement with the idea of an atomic nucleus “as a gas of interacting quasiparticles”.⁶ The magnitude of shell effects decreases with increasing atomic numbers, primarily because the magnitude of the magic gap falls off rapidly in heavy and superheavy nuclei. With merging phenomenon coming into play, the situation changes. For one thing, merging ensures a negative contribution to the ground-state energy, despite the fact that the density of the single-particle levels exhibits an opposing trend, reaching a maximum at the Fermi surface, rather than a minimum, as in the conventional shell model.

In effect, the existence of gaps in filling prevents the merging phenomenon from working in full force. However, in heavy and superheavy atomic nuclei where this gap becomes smaller and smaller, the FC correction to the Weizsäcker ground state-energy grows in importance, creating the opportunity for level merging to be an important playwright in the formation of new stability islands.

C. Merging of single-particle levels in an external magnetic field

In this subsection, we employ the simplest model of fermion condensation, proposed in Ref. 3, to investigate the impact of an external magnetic field on the FC spectrum and demonstrate that the presence of this field does not prevent the Fermi surface from swelling. The analysis is restricted to the simplest form of the FL energy functional, namely

$$E = \frac{1}{2M} \int \tau(\mathbf{r}) d\mathbf{r} + \frac{1}{2} \iint f(\mathbf{r} - \mathbf{s}) \rho^2(\mathbf{r}, \mathbf{s}) d\mathbf{r} d\mathbf{s}, \quad (48)$$

in which f stands for the Landau interaction function. The density matrix ρ and kinetic-energy density τ are respectively given by

$$\rho(\mathbf{r}, \mathbf{s}) = \sum n_i \Psi_i^*(\mathbf{r}) \Psi_i(\mathbf{s}), \quad \tau(\mathbf{r}) = \sum n_i \nabla \Psi_i^*(\mathbf{r}) \nabla \Psi_i(\mathbf{r}), \quad (49)$$

in terms of the true quasiparticle occupation numbers n_i of the single-particle states. Imposition of the external magnetic field \mathbf{H} leads to a set of coupled equations

$$\frac{(\mathbf{p} - e\mathbf{A}(\mathbf{r}))^2}{2M} \Psi_j(\mathbf{r}) + \int f(\mathbf{r} - \mathbf{s}) \rho(\mathbf{r}, \mathbf{s}) \Psi_j(\mathbf{s}) d\mathbf{s} = E_j \Psi_j(\mathbf{r}), \quad (50)$$

for the eigenvalues E_j and eigenfunctions Ψ_j , where \mathbf{A} is the corresponding vector potential. Specifically, we address 2D homogeneous electron systems, employing the Landau gauge in which $A_x = -Hy$ and $A_y = 0$. Generally, each of the equations (50) contains a sum of an infinite number of terms involving different wave functions Ψ_i . This complication is normally surmounted by working within a quasiclassical approximation,³⁹ such that

$$\rho(\mathbf{r}, \mathbf{s}, \mathbf{A}) = e^{i\mathbf{A}(\mathbf{r})(\mathbf{r}-\mathbf{s})} \rho_0(\mathbf{r} - \mathbf{s}), \quad (51)$$

where ρ_0 is the well-known quasiparticle density matrix for homogeneous matter. However, the quasiclassical approximation fails in systems having long-range interaction functions of the kind responsible for flattening of the single-particle spectrum $\epsilon(p)$ beyond the QCP.

On the other hand, it is just the long-range character of the relevant effective interactions that allows one to proceed while avoiding the approximation (51). Indeed, suppose the correlation radius ξ specifying the behavior of f at large distances is in excess of the Larmor radius $r_H = 1/\sqrt{eH}$. In the sum over i implicit in Eq. (50), the diagonal term is dominant, and the initial structure $\Psi(x, y) = e^{ip_x x} \chi(y)$ of the corresponding wave functions is recovered. Neglecting corrections of order r_H^2/ξ^2 coming from nondiagonal contributions, we are then left with the Landau-like equation

$$\frac{1}{2M} (p_y^2 + e^2 H^2 (y - y_0)^2) \chi_j(y) = (E_j - f n_j) \chi_j(y), \quad (52)$$

where f is the effective coupling constant. As usual, y_0 specifies the location of the Larmor circle, while the occupation number $n_j = [1 + \exp(\epsilon_j/T)]^{-1}$ is itself dependent upon the deviation $\epsilon_j = E_j - \mu$ of the single-particle level from the chemical potential μ . Now, according to Eq. (52), the initial equidistant spectrum $E_j^0 \equiv E_j(H = 0) = \omega_c^0(j + 1/2)$, with $\omega_c^0 = eH/M$, gives way to a modified level scheme determined by integer solutions of nonlinear algebraic equation

$$\epsilon(x) = \omega_c^0 x - \mu + f n(x). \quad (53)$$

Recalling that $n(\epsilon = 0) = 1/2$, Eq. (53) may be recast in the form

$$n(\epsilon) = \frac{\epsilon}{f} + \frac{1}{2} \left(1 - \frac{y}{r}\right), \quad (54)$$

where $y = x - (\mu - f/2)/\omega_c^0$ gives the displacement of the single-particle levels from the Fermi surface. Eq. (54) is reminiscent of the corresponding equation for the NFL occupation numbers in the Nozières model,³ developed for homogeneous matter with an interaction function $f(\mathbf{r} - \mathbf{s}) = \text{const.}$ of infinite radius. The QCP of the model³ is located at $f = 0$, with the flattening phenomenon in effect at $f > 0$.

Eq. (54) can easily be solved graphically at $T = 0$ (see Fig. 3). This is done by plotting the kink $n(\epsilon)$, together with the set of straight lines representing the right side

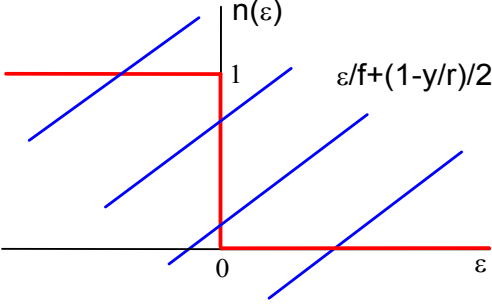


FIG. 3: Graphical illustration of the solution of Eq. (54)

of Eq. (54), against the input parameter y . At $y > r$, the right side of Eq. (54) has the form $\epsilon - a$, with $a > 0$. In this case, crossing points lie in the right half-plane $\epsilon > 0$ where $n = 0$, implying that the initial Landau spectrum $E_j^0 = \omega_c(j+1/2)$ is unaffected by the interactions. On the other hand, at $y < -r$, the crossing points transfer to the left half-plane $\epsilon < 0$ where $n = 1$, so that the equidistant Landau spectrum is simply shifted by the constant f .

What happens in the interval $-r < y < r$? In this case, as seen from Fig. 3, the crossing points land on the vertical segment of the kink, yielding the solution $\epsilon = 0$; Eq. (54) reduces to relation

$$n_*(y) = \frac{1}{2} \left(1 - \frac{y}{r} \right), \quad -r < y < r, \quad (55)$$

for finding the NFL occupation numbers $n_*(y, H)$ at $H \neq 0$. Comparison with the corresponding result $n_*(p) = (\mu - p^2/(2M))/f$ for the Nozières model³ shows that the swelling of the Fermi surface inherent in this model persists, the range of the FC domain being unaffected by imposition of the magnetic field. At finite T , analysis of Eq. (54) confirms that in the FC domain, the derivative $d\epsilon(x)/dx$ becomes proportional to T , again in accordance with the results obtained in homogeneous matter at $H = 0$.³

VII. LIFSHITZ PHASE DIAGRAM: EVOLUTION FROM THE LANDAU STATE TO THE FC STATE

Low-temperature phase transitions occurring in Fermi systems have already been studied for a hundred years. As a rule, these are second-order phase transitions, epitomized in the phenomenon of superconductivity discovered by Kamerlingh-Onnes in 1911. By contrast, experimental studies of topological phase transitions (TPTs) have gained prominence only during the last decade. This disparity stems from the fact that low-temperature, second-order phase transitions associated with violation of one or another of the Pomeranchuk stability conditions may arise in weakly or moderately correlated systems, as in the example of superconductivity. In the case

of TPTs, it is instead the necessary stability condition (20) for the Landau state that suffers a breakdown, an eventuality that can be realized only in strongly correlated Fermi liquids, notably in dense films of liquid ³He and the dilute 2D electron liquid in MOSFETs. Nothing was known about such systems prior to recent experimental and technological advances. To our knowledge, the possibility of a many-fermion ground state with a multi-connected Fermi surface was first entertained by Frölich in 1950.⁶⁸ Theoretical consideration of topological aspects of the electron liquid in solids dates back to an article by I. M. Lifshitz published ten years later.⁶⁹

The Lifshitz analysis requires specific information on the structure of the single-particle spectrum $\epsilon(\mathbf{p})$, which is often not available because this spectrum may be altered dramatically upon switching on the effective interactions between quasiparticles.¹ For many years, insights into the effects of these interactions were pursued within the Hartree-Fock framework.^{70–73} It was not until 1990 that such limited descriptions began to be replaced by more powerful FL approaches.^{1–3,16–19,24–27,74}

In dealing with homogeneous systems, any TPT is associated with a change in the number of roots of equation⁶⁴

$$\epsilon(p; n, \rho) = 0, \quad (56)$$

where $\epsilon(p; n)$ is the spectrum of single-particle excitations measured from the Fermi surface and evaluated self-consistently with the $T = 0$ ground-state quasiparticle momentum distribution $n(p)$. In conventional homogeneous Fermi liquids such as 3D liquid ³He, Eq. (56) has a single root, the Fermi momentum p_F ; hence the corresponding ground-state quasiparticle momentum distribution is simply $n_F(p) = \theta(p_F - p)$.

The QCP corresponds to the particular case of a topological transition for which the bifurcation point p_b coincides with the Fermi momentum p_F . By its definition, at the QCP the effective mass diverges and therefore the Fermi velocity $v_F(\rho_\infty) = (d\epsilon(p, \rho_\infty)/dp)_0$ vanishes. In this situation, the spectrum is constrained to have an inflection point,²² i.e. two additional roots of Eq. (56) emerge simultaneously at the Fermi surface, implying that $\epsilon(p) \propto (p - p_F)^3$ at the QCP itself. It will be seen that such a situation is not typical. As a matter of fact, as the system moves into the quantum critical regime from the FL domain, Eq. (56) must *already* acquire two new roots $p_1, p_2 \neq p_F$ at some critical value λ_{QCP} of the input parameter λ . Accordingly, the Fermi surface becomes multi-connected, but with the ground-state momentum distribution $n(p) = \theta(p_1 - p) - \theta(p_2 - p) + \theta(p_F - p)$, still satisfying the usual FL relation $n^2(p) = n(p)$. On the other hand, those solutions of Eq. (6) emerging at some critical coupling λ_{FC} and featuring a completely flat portion of the spectrum $\epsilon(p)$ correspond to solutions of Eq. (56) having an *infinite* number of roots, or equivalently, solutions with an infinite number of sheets of the Fermi surface. We reiterate that in the associated FC phase, the fundamental relation $n^2(p) = n(p)$ of stan-

standard FL theory no longer holds; *partial occupation* of single-particle states must occur.

Numerous calculations have demonstrated that as a rule, the ratio $\lambda_{FC}/\lambda_{QCP}$ lies between 1 and 2. When the coupling constant λ increases beyond λ_{QCP} into the regime of instability of the Landau state, a striking sequence of rearrangements of the single-particle degrees of freedom is witnessed,¹⁷ as pockets of the Fermi surface breed and proliferate rapidly, in resemblance to a fractal cascade. Our first task in the present section is to examine the details of the evolution of the topology of the ground state in the Lifshitz phase diagram over the coupling interval $\lambda_{QCP} < \lambda < \lambda_{FC}$. Our second goal is to expose the intricate evolution, with increasing temperature, of the single-particle spectrum $\epsilon(p)$ in phases with having a multi-connected Fermi surface.

A. Generic features of topological transitions in Fermi liquids

In principle, the bifurcation p_b of Eq. (56) giving rise to the first TPT can emerge at any point of momentum space. Leaving aside the case addressed in the previous section, in which the transition leads to a swollen Fermi surface, we are left with the following options. The first and common option is $p_b = 0$. The second and particular one is the QCP for which $p_b = p_F$. In the third case $0 < p_b < p_F$, the bifurcation point p_b is a local maximum of the spectrum $\epsilon(p)$, since one has $\epsilon(p) \leq 0$ throughout the Fermi volume. This implies that between p_b and p_F there also exists a local minimum of $\epsilon(p)$, because by definition the spectrum $\epsilon(p)$ changes sign at $p = p_F$. Thus, if the bifurcation point is located between 0 and p_F , the equation $d\epsilon(p)/dp = 0$ has *at least* two roots in this momentum interval. Simple analysis demonstrates that existence of at least two roots of this equation is also inherent in the fourth and last case, in which Eq. (56) bifurcates at a momentum greater than p_F and the bifurcation point is a local minimum of the curve $\epsilon(p)$.

A prerequisite for the occurrence of the bifurcations in Eq. (56) is a marked momentum dependence of the interaction function $f(\mathbf{p}, \mathbf{p}_1)$. Such a momentum dependence is often associated with forces of long range in coordinate space that are induced in advance of an impending second-order phase transition. As a pertinent illustration, let us consider the situation in the vicinity of pion condensation, a phase transition in dense neutron matter predicted by A. B. Migdal.^{75,76} We restrict ourselves to the case of moderate pion fluctuations where the Ornstein-Zernike approximation, allowing convenient treatment of the fluctuation contribution to the interaction function f , is still applicable. In this case, the exchange contribution $f_{\pi\pi}$ to the interaction function f has the form⁷⁵

$$f_{\pi\pi}(q) = \frac{g}{\kappa^2(\rho) + (q^2/q_c^2 - 1)^2}, \quad (57)$$

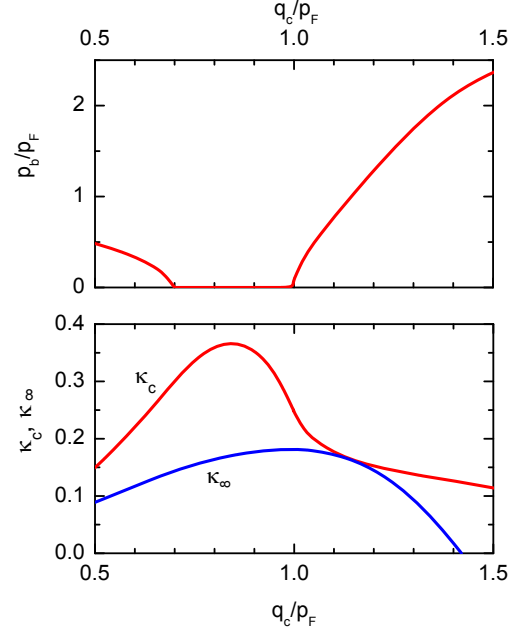


FIG. 4: Upper panel: Position of the bifurcation point p_b in units of the Fermi momentum, versus the critical wave number q_c (also in units of p_F). Lower panel: critical parameters κ_b and κ_∞ as functions of q_c/p_F .

with coupling constant g and the critical momentum taken as $q_c \simeq (0.7 \div 1.0) p_F$. The stiffness coefficient $\kappa^2(\rho)$ vanishes at the critical density ρ_c , specifying the point where pion condensation sets in. For simplicity, we choose $g = 1/(2m_\pi^2)$, corresponding to bare πNN vertices, and study the behavior of quantities versus $\kappa(\rho)$.

In the Landau state, corrections to the neutron spectrum $\epsilon(p)$ associated with the exchange of pion fluctuations are evaluated in closed form by means of Eq. (6), based on the quasiparticle distribution $n_F(p)$. Substituting the expression (57) and integrating, we obtain

$$\frac{d\epsilon(p)}{dp} = \frac{p}{M} + \frac{gq_c^4}{16\pi^2 p^2} \left(-\frac{1}{2} L(p) + \frac{p^2 + p_F^2 - q_c^2}{\kappa q_c^2} A(p) \right), \quad (58)$$

where

$$L(p) = \ln \frac{[(p+p_F)^2 - q_c^2]^2 + \kappa^2 q_c^4}{[(p-p_F)^2 - q_c^2]^2 + \kappa^2 q_c^4} \quad (59)$$

and

$$A(p) = \arctan \frac{(p_F + p)^2 - q_c^2}{\kappa q_c^2} - \arctan \frac{(p_F - p)^2 - q_c^2}{\kappa q_c^2}. \quad (60)$$

Further integration can be performed analytically,¹⁹ but the results are too cumbersome to present here.

Results of numerical calculations displayed in the upper panel of Fig. 4 demonstrate that a new root of Eq. (56), lying at the origin, appears already at $\kappa = \kappa_b \simeq 0.4$, signaling that the Landau state has become

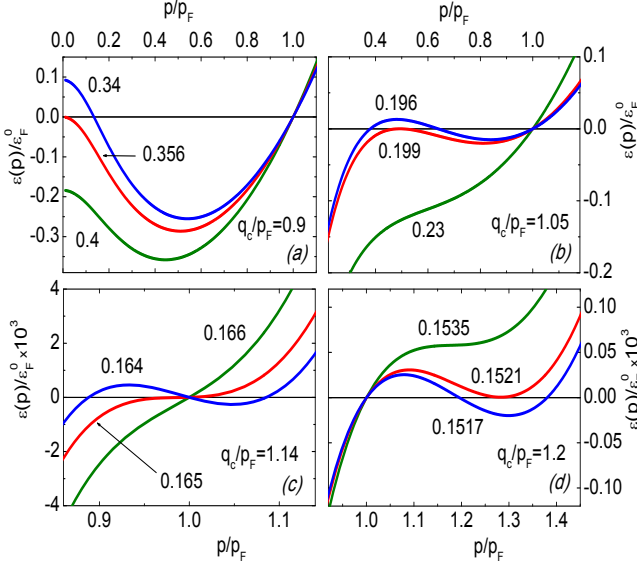


FIG. 5: Neutron quasiparticle spectra $\epsilon(p)$ (in units of ϵ_F^0) evaluated for $q_c = 0.9 p_F$ (panel (a)), $q_c = 1.05 p_F$ (panel (b)), $q_c = 1.14 p_F$ (panel (c)), and $q_c = 1.2 p_F$ (panel (d)). Corresponding values of the parameter κ are indicated near the curves.

unstable *well before* the system attains the point of pion condensation. At customary values of the critical momentum q_c , one has $p_b = 0$. However, as q_c increases to greater values, p_b rapidly moves toward the Fermi momentum and crosses the Fermi surface at $q_c \simeq 1.14 p_F$. The dependence of the critical parameter κ_b on the wave number q_c , depicted in the lower panel, shows that the largest values of κ_b are achieved just in the preferred range $q_c/p_F \sim 0.7 - 1.0$. The value κ_∞ of κ at which the border of the instability region reaches the Fermi momentum p_F is also plotted in the lower panel of Fig. 4. The resulting curve lies below the curve of $\kappa_b(q_c)$ everywhere except for the point of contact at $q_c \simeq 1.14 p_F$.

The corresponding evolution of the neutron spectra is illustrated by panels (b)–(d) of Fig. 5 where the spectra $\epsilon(p)$ calculated for $q_c = 1.05 p_F$, $1.14 p_F$, and $1.2 p_F$ are drawn. We see that at $q_c < p_F$ the bifurcation point p_b does emerge quite close to the origin. This topological transition plays important role in the acceleration of neutrino cooling of neutron stars,⁷⁷ since the presence of a new pocket of the Fermi surface in the low-momentum region lifts the ban on neutron β -decay and activates the highly efficient direct-Urca process.

As seen from panels (d) and (e) of Fig. 5, beyond the Fermi liquid QCP the curve $\epsilon(p)$ crosses the Fermi level three times, the $T = 0$ Landau momentum distribution $n_F(p) = \theta(p_F - p)$ is rearranged, and the Fermi surface becomes multi-connected. In this instance, where there is a single bubble in the filling pattern, the new $T = 0$ occupation numbers $n(p, T = 0)$ are given by an alternating sequence of two numbers, 1 and 0. The occupancy

is $n(p) = 1$ at $p < p_1$, while at $p_1 < p < p_2$, there is a gap in filling where $n(p) = 0$, and at $p_2 < p < p_3$, the occupancy is once again $n(p) = 1$. This is a typical topological phase transition, at which no symmetry inherent in the ground state is violated.

B. Prerequisites for the proximity of the first TPT and the QCP

Interest to the mechanisms underlying violation of FL theory has been sparked by the discovery of NFL behavior of electron systems present in numerous heavy-fermion metals. It is a specific and significant feature of these NFL phenomena that the bifurcation point p_b lies close to the Fermi momentum p_F . To clarify the conditions necessary for the proximity of p_b and p_F , it is natural to begin with the Taylor expansion of the group velocity,

$$v(p) = v_F + v_1 \frac{p - p_F}{p_\infty} + \frac{1}{2} v_2 \frac{(p - p_F)^2}{p_F^2}, \quad (61)$$

applicable in the immediate vicinity of the Fermi surface, which yields the spectrum

$$\epsilon(p) = v_F(p - p_F) + \frac{1}{2p_F} v_1 (p - p_F)^2 + \frac{1}{6p_F^2} v_2 (p - p_F)^3. \quad (62)$$

Upon inserting this expression into Eq. (56), we observe (once again) that *already* on the disordered side of the QCP where $v_F(\rho)$ is still *positive*, this equation picks up two additional real roots:

$$p_{1,2} - p_F = -p_F \frac{3v_1}{2v_2} \left(1 \pm \sqrt{1 - \frac{8v_F(\rho)v_2}{3v_1^2}} \right). \quad (63)$$

Accordingly, the first TPT occurs at a critical density ρ_t where the Fermi velocity is given by $v_F(\rho_t) = 3v_1^2/8v_2$. This analysis is self-consistent as long as the ratio v_1/v_2 is small; otherwise at least one of the roots lies too far from the Fermi momentum for the Taylor expansion (61) to be valid. In conventional Fermi liquids, where $f(\mathbf{p}, \mathbf{p}_1)$ is a smooth function in momentum space and hence the coefficients v_1 and v_2 are of the same order, the stated condition is difficult to satisfy. However, the situation becomes more favorable in the vicinity of a second-order phase transition where the fluctuations are of long wavelength. Assuming these fluctuations to be rather weak, their contribution \mathcal{F}^{sf} to the interaction function f can be accounted for within well-known Ornstein-Zernike (OZ) approximation to yield

$$\mathcal{F}_{\alpha\beta\gamma\delta}^{\text{sf}}(\mathbf{p}, \mathbf{p}_1) \rightarrow g^2 \sigma_{\alpha\beta} \sigma_{\gamma\delta} \chi(|\mathbf{p} - \mathbf{p}_1|). \quad (64)$$

Here g is the spin-fluctuation vertex and $\chi(q) = 4\pi/(q^2 + \xi^{-2})$ is the spin susceptibility, ξ being the correlation radius, which diverges at the second-order phase transition point. If the spin-fluctuation term is well-pronounced,

then ξ substantially exceeds the distance between particles. The applicability of the OZ approximation requires that the inequality $(g^2/\pi v_F^0) \ln(p_F \xi) \leq 1$ should be met.²⁷

The spin-fluctuation contribution $v^{\text{sf}}(p)$ to the group velocity $v(p)$ is evaluated straightforwardly, employing the identity $\sigma_{\alpha\beta}\sigma_{\gamma\delta} = \frac{3}{2}\delta_{\alpha\delta}\delta_{\gamma\beta} - \frac{1}{2}\sigma_{\alpha\delta}\sigma_{\gamma\beta}$ to obtain

$$v^{\text{sf}}(p) = \frac{3g^2}{2\pi} \ln \frac{1 + \xi^2(p - p_F)^2}{4p_F^2 \xi^2}. \quad (65)$$

Then instead of Eqs. (61) and (62) one obtains (with $v_F^0 = p_F/M$ and $v_2 = \lambda_s v_F^0$)

$$v(p) = \frac{p_F}{M^*} + v_1 \frac{p - p_F}{p_F} + \frac{3}{2} \lambda_s v_F^0 \frac{(p - p_F)^2}{p_F^2},$$

$$\epsilon(p) = \frac{p_F}{M^*} (p - p_F) + v_1 \frac{(p - p_F)^2}{2p_F} + \lambda_s v_F^0 \frac{(p - p_F)^3}{2p_F^2}, \quad (66)$$

where the effective mass M^* is given by

$$\frac{M}{M^*} = \frac{M}{M_{\text{FL}}^*} - 3 \frac{g^2}{\pi v_F^0} \ln(2p_F \xi(\rho)), \quad (67)$$

M_{FL}^* being the FL effective mass evaluated without inclusion of the fluctuation contribution (which drives the system toward the QCP), while

$$\lambda_s = \xi^2 p_F^2 \frac{g^2}{\pi v_F^0} \simeq \frac{\xi^2 p_F^2}{\ln(p_F \xi)} \gg 1. \quad (68)$$

Since the condition $v_2/v_1 \propto 1/\lambda_s \ll 1$ holds in the present case, the proximity of the first TPT bifurcation point to the Fermi momentum is established.

In addition, the density ρ_∞ corresponding to the topological QCP, which signals instability of the Landau state, is determined by condition

$$\frac{M}{3M_{\text{FL}}^*(\rho_\infty)} = \frac{g^2}{\pi v_F^0} \ln(2p_F \xi(\rho_\infty)). \quad (69)$$

In strongly correlated Fermi systems, the ratio M/M_{FL}^* is suppressed, implying that the stability condition (69) is violated *well before* the correlation radius $\xi(\rho)$ grows to values sufficient to put the OZ approximation in jeopardy.

It is worth estimating the limits of the applicability of the Taylor expansion (66). As seen from Eq. (65), the expansion breaks down in case the difference $p - p_F$ reaches values of order of $1/\xi$. At larger distances from the Fermi surface, the logarithmic term in Eq. (65) asserts itself, introducing logarithmic corrections to the specific heat and other thermodynamic quantities (as shown explicitly below).

C. Cascade of topological transitions in the QCP region

The emergence of new small pockets of the Fermi surface is an integral feature of the QCP phenomenon,

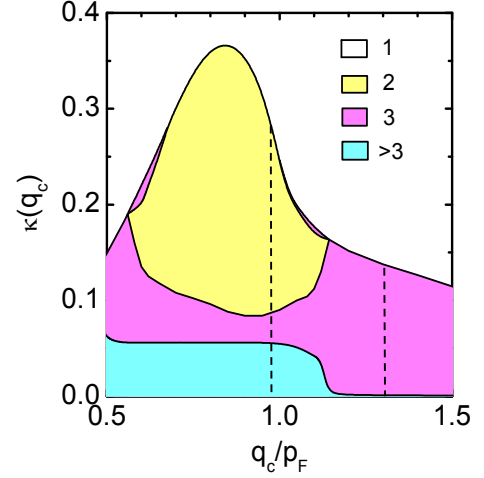


FIG. 6: The phase diagram of neutron matter in (q_c, κ) variables. The Landau phase (FL) occupies the white region of the plane. The index i indicates the number of sheets (or pockets) of the Fermi surface.

whether one deals with 2D liquid ^3He , high- T_c superconductors, or strange heavy-fermion metals. A striking peculiarity of the segment of the Lifshitz phase diagram beyond the first TPT point, revealed in Refs. 17, is the fast breeding of these pockets as λ increases beyond λ_{QCP} . This process is illustrated in Fig. 6, which shows theoretical results¹⁹ for the phase diagram of neutron matter in the (q_c, κ) plane, from calculations performed for the interaction function (57) in the vicinity of the threshold of pion condensation. The Landau state with $n(p) = n_F(p)$ occupies the white region of the diagram (labeled FL in the figure), while the phases exhibiting new pockets of the Fermi surface populate the shaded portion of the plane, which is separated from the FL domain by the curve $\kappa_b(q_c)$. Ref. 20 reports interesting results from a simple model incorporating feedback of fermion condensation on critical fluctuations. In particular, the range of territory on the phase diagram in which the FC wins the contest with the multi-pocket phase was estimated as $0.95 < q_c/p_F < 1.3$ at $\kappa \simeq 0.1$.

As seen in Figs. 7 and 8, the breeding phenomenon also expresses itself vigorously in a different model¹⁷ based on the interaction function $f(q) = g/(q^2 + \kappa^2)$.

The occurrence such cascades is associated with the condition $n(p) \leq 1$ enforced by the Pauli principle. As we have seen earlier, solutions of the equilibrium equation (18) always exist, but these solutions violate the restriction $n(p) \leq 1$ until the coupling constant λ attains a critical value λ_{FC} . It is this restriction that triggers a cascade of TPTs in the segment of the Lifshitz phase diagram defined by $\lambda_{\text{QCP}} < \lambda < \lambda_{\text{FC}}$. Indeed, in the TF problem where no such restriction is present, no topological transitions exist. Details of the transitions between phases with a multi-connected Fermi surface and phases hosting a FC remain unclear at this point.

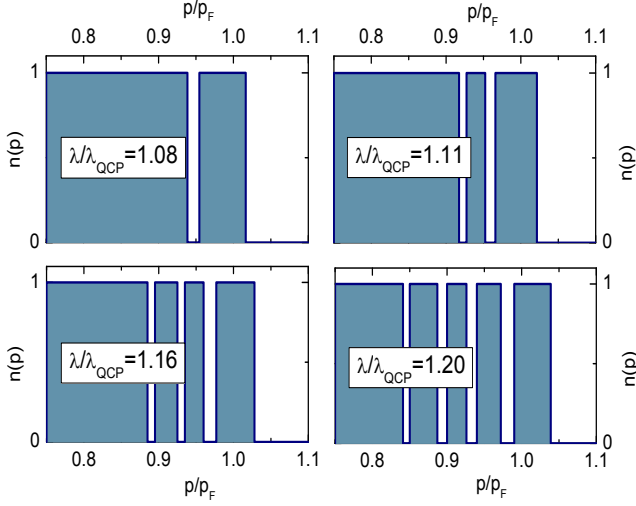


FIG. 7: Cascade of topological transitions, evaluated in the model with $f(q) = \lambda(q^2 + \kappa^2)^{-1}$ with $\kappa = 0.07 p_F$ for four values of $\lambda/\lambda_{\text{QCP}} = 1.08, 1.11, 1.16, 1.20$.

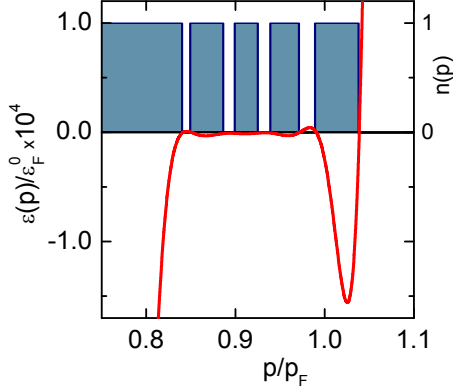


FIG. 8: Single-particle spectrum $\epsilon(p)$ in units of $\epsilon_F^0 = p_F^2/2M$ together with the self-consistent quasiparticle momentum distribution $n(p)$, as evaluated in Ref. 19 within a model in which the Landau interaction function has the form $f(q) = \lambda(q^2 + \kappa^2)^{-1}$, with $\kappa = 0.07 p_F$. Calculated for an effective coupling constant λ , exceeding the critical QCP value λ_{QCP} by 30%, this spectrum possesses nine nodes, corresponding to four hole pockets in momentum space.

In this section, we discuss results of numerical calculations of single-particle spectra $\epsilon(p, T)$ and quasiparticle momentum distributions $n(p, T)$ beyond the first TPT.²⁶ As we have seen, (i) these transitions are associated with the violation of the necessary stability condition (20) for the Landau state at $T = 0$, in which $n(p) = \theta(p_F - p)$, and (ii) there exist different alternatives for the rearrangement of the Landau state beyond the transition point. The first possibility is the emergence of a small pocket of the Fermi surface; hence, the new quasiparticle momentum distribution $n(p, T = 0) = \theta(p_1 - p) - \theta(p_2 - p) + \theta(p_F - p)$ features a combination of

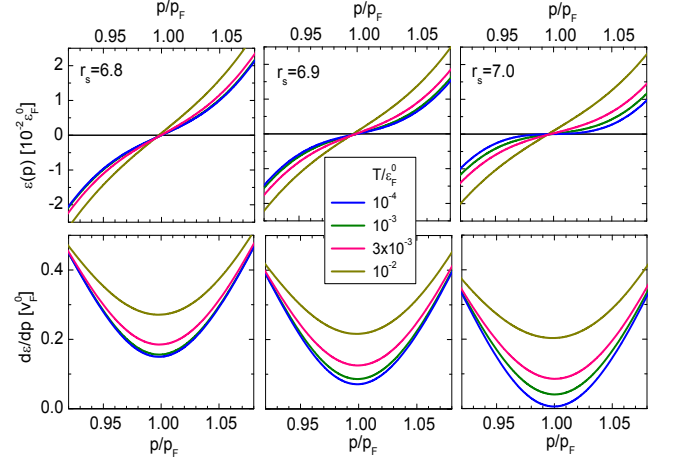


FIG. 9: Single-particle spectrum $\epsilon(p)$ (top panels) and derivative $d\epsilon(p)/dp$ in units of $v_F^0 = p_F/M$ (bottom panels), evaluated in the model (70) with parameters $g_2 = -0.16$ and $\beta_2 = 0.14$, chosen to adequately describe results of microscopic calculations⁵⁰ of 2D-electron-gas single-particle spectrum at $T = 0$, are shown as functions of p/p_F at four (line-coded) temperatures expressed in units of $10^{-2} \epsilon_F^0$ at $r_s = 6.8$ (left column), $r_s = 6.9$ (middle column), and $r_s = 7.0$ (right column).

three kinks. The second possibility amounts to a swelling of the Fermi surface, so that the ne momentum distribution $n(p, T = 0)$ is continuous over a momentum interval $[p_1, p_2]$ embracing the boundary momentum p_F . In this momentum interval, the spectrum $\epsilon(p, T)$ is completely flat at $T = 0$, and, according to Eq. (31), it is inclined at finite T with a slope proportional to T .

On the other hand, in the state having a multi-connected Fermi surface, the spectrum $\epsilon(p, T = 0)$ varies smoothly in the space beyond the sheets of the Fermi surface but oscillates rapidly in the space between them. The magnitude of the departure of $|\epsilon(p, T = 0)|$ from 0 in this domain, denoted below by T_m , emerges as a new energy scale of the problem, at which “melting” of the structure associated with the well-defined multi-connected Fermi surface can occur. Indeed, as seen from the basic Landau formula (4), the distribution $n(p, T = 0)$ remains almost unchanged as long as $T < T_m$. However, as the temperature T increases, kinks in $n(p)$ become smeared, and at $T > T_m$, in the momentum domain between the sheets, the function $n(p, T)$ becomes continuous and almost T -independent. In this case, according to Eq. (31), the dispersion of the single-particle spectrum $\epsilon(p)$ becomes proportional to T . Thus at $T > T_m$, the temperature evolution of $\epsilon(p, T)$ and $n(p, T)$ becomes universal.

To provide tangible support for this important conclusion, Figs. 9–13 present the results of model calculations of the spectrum $\epsilon(p, T)$ and momentum distributions $n(p)$ performed in Ref. 26 on the basis of Eq. (6). Two different interaction functions f are employed in these calcu-

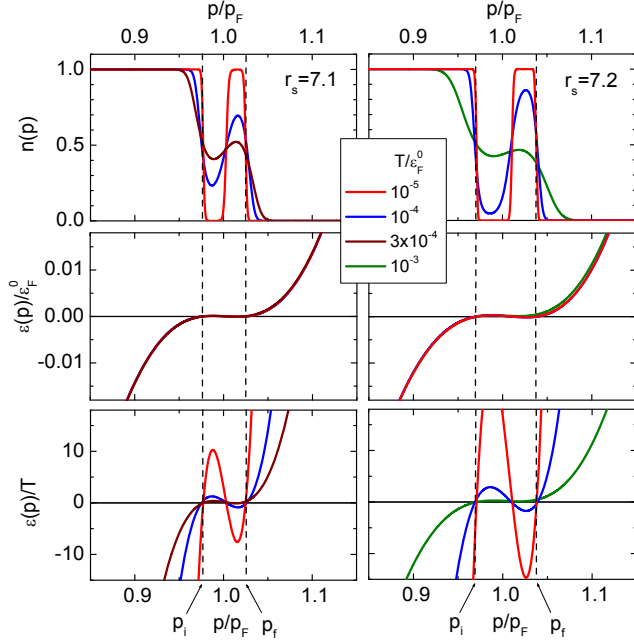


FIG. 10: Occupation numbers $n(p)$ (top panels), single-particle spectrum $\epsilon(p)$ in units of ϵ_F^0 (middle panels), and ratio $\epsilon(p)/T$ (bottom panels) for the model (70) at $r_s = 7.1$ (left column) and $r_s = 7.2$ (right column), exceeding the QCP value $r_\infty = 7.0$, evaluated with the same set of the parameters as in Fig. 9. All three quantities are shown as functions of p/p_F at different temperatures in units of ϵ_F^0 , lower than the transition temperature $T_m = 10^{-3}\epsilon_F^0$.

lations, namely

$$f(k) = g_2 \frac{\pi}{M} \frac{1}{(k^2/4p_F^2 - 1)^2 + \beta_2^2}, \quad (70)$$

with $g_2 = -0.16$ and $\beta_2 = 0.14$ and

$$f(k) = g_3 \frac{\pi^2 p_F}{M} \frac{1}{k^2 + \beta_3^2 p_F^2}. \quad (71)$$

The first is chosen because this function adequately reproduces, within the extended Landau-Migdal treatment, the results of microscopic calculations⁵⁰ of the single-particle spectrum $\epsilon(p)$ of the 2D electron gas at $T = 0$, and because the corresponding initial TPT essentially coincides with the QCP located at $r_s = r_\infty = 7.0$. The second choice of f is relevant to a 3D system with spin fluctuations and has already been considered above in Sec. VII.B.

Fig. 9 displays results for the single-particle spectrum $\epsilon(p)$ and the group velocity $d\epsilon(p)/dp$, obtained on the FL side of the QCP for the model (70). It is seen that as the QCP is approached, the group velocity becomes a parabolic function of momentum p . With increasing r_s its minimum gradually approaches the horizontal axis, touching down for $r_s = r_\infty$ exactly at the point $p = p_F$, so that $\epsilon(p, T = 0) \sim (p - p_F)^3$ has an inflection point

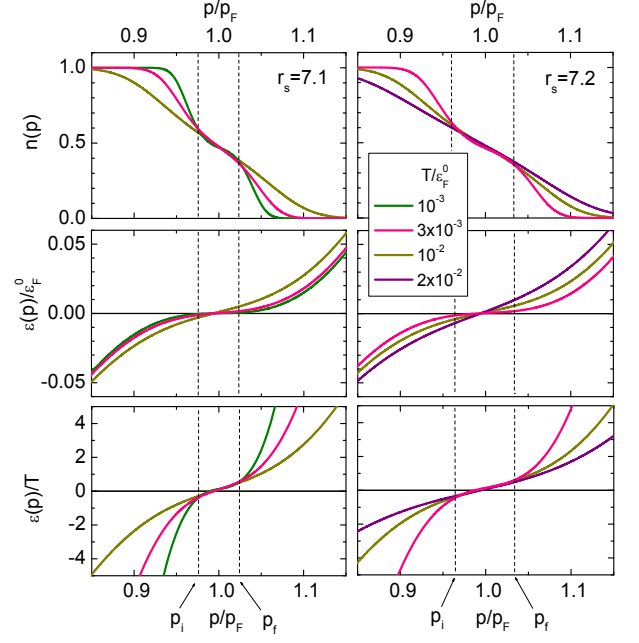


FIG. 11: Same as in Fig. 10 but at temperatures, higher than T_m .

at the Fermi surface. At $r_s > r_\infty$, the group velocity v_F evaluated with $n(p) = n_F(p)$ changes its sign, triggering the rearrangement of the Landau state, in accordance with the topological scenario for the QCP.

We turn now to Figs. 10 and 11, which show numerical results for the spectrum $\epsilon(p)$ and quasiparticle momentum distribution $n(p)$, calculated for the interaction function (70) both below and above the temperature T_m . It should be noted that the value of T_m is somewhat uncertain, since we are dealing here with a crossover rather than a well-defined phase transition. Nevertheless, comparison of these two figures reveals a striking alteration of the structure of both $\epsilon(p)$ and $n(p)$ upon passage across T_m from the lower to the higher temperature. Indeed, as seen from the top panel of Fig. 10, a well-defined multi-connected Fermi surface, distinguished by a pronounced gap in filling, exists only at extremely low temperatures $T < 10^{-4}\epsilon_F^0$, while at $T \simeq T_m \simeq 10^{-3}\epsilon_F^0$, the gap in the occupation numbers closes, although at $T < T_m$, some T -dependence of $n(p)$ in the vicinity of the Fermi surface is still present. On the other hand, as seen from the top panel of Fig. 11, a T -independent behavior of the momentum distribution $n(p)$, existing at $T > T_m$ in the FC domain, holds over a wide temperature interval. Comparison of the third panels of Figs. 10 and 11 demonstrates that huge variations of the ratio $\epsilon(p)/T$, existing at $T < T_m$, completely disappear in the FC domain at $T > T_m$.

The same conclusions follow from Figs. 12 and 13, which display the temperature behavior of the spectrum $\epsilon(p)$ and momentum distribution $n(p)$, evaluated for the interaction function (71) below and above T_m .

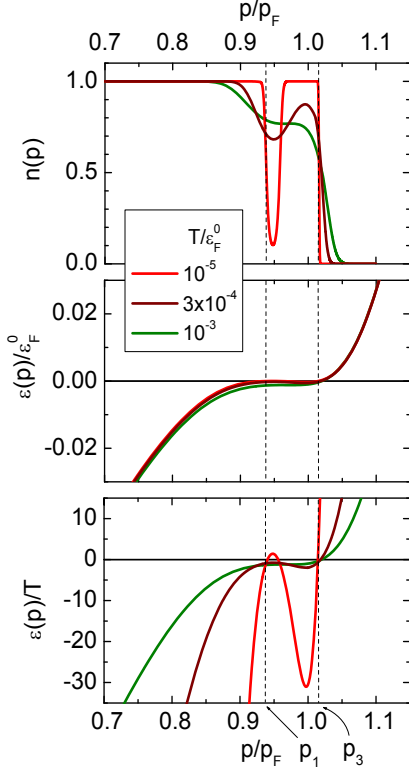


FIG. 12: Occupation numbers $n(p)$ (top panel), single-particle spectrum $\epsilon(p)$ in units of $10^{-3} \epsilon_F^0$ (middle panel), and ratio $\epsilon(p)/T$ (bottom panel), plotted versus p/p_F at three line-type-coded temperatures in units of ϵ_F^0 , taken below the transition temperature $T_m = 3 \times 10^{-3} \epsilon_F^0$. The model (71) is assumed with parameters $g_3 = 0.45$ and $\beta_3 = 0.07$.

It is instructive to compare the results shown in Figs. 10–13 with corresponding results derived for the singular interaction function

$$f(k) = g_* \frac{\pi^2}{M} \frac{e^{-\beta_* k/p_F}}{k}, \quad (72)$$

studied in Ref. 16 as a model of FC analytically soluble at $T = 0$. (All the input parameters in the interactions defined in Eqs. (70), (71), and (72) are dimensionless.)

It is characteristic of systems with a singular interaction function (singular as $k \rightarrow 0$, hence of long range in coordinate space) that the linear-in- T dispersion of the single-particle spectrum symptomatic of the FC domain exists at any temperature T . This behavior is evident in the numerical results for the interaction model (72) plotted Fig. 14.

From these and other numerical and analytical studies, it may be concluded that specific features of the flattening of single-particle spectra beyond the first TPT point are *universal*, differing only with respect to the scale temperature T_m .

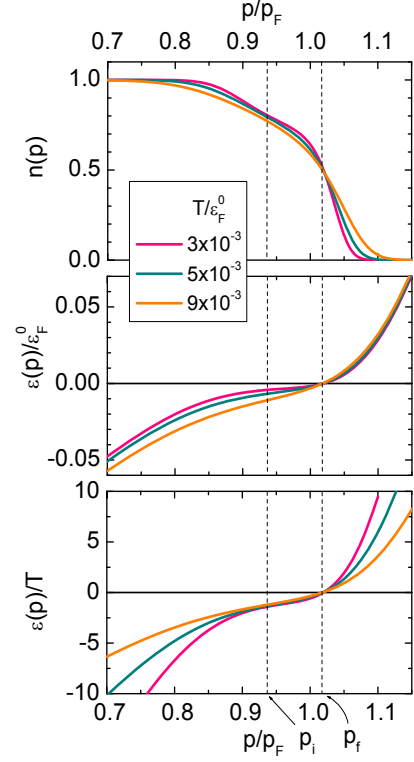


FIG. 13: Same as in Fig. 12 but at temperature higher than T_m .

VIII. UNCONVENTIONAL THERMODYNAMICS OF SYSTEMS WITH A FC

In this section we will focus on certain unusual features of the thermodynamic properties of strongly correlated systems hosting a FC. In making concrete estimations, it will be assumed that the interaction function has a small, specific fluctuation contribution, the origin and form of which has been analyzed in Sec. VII.B.

A. Logarithmic corrections to $\gamma(T)$

The standard FL behavior of $\gamma(T \rightarrow 0) = \text{const.}$ does in fact re-emerge beyond the point of a TPT, because the density of states $N(0)$ again becomes finite. However, recovery of FL behavior occurs only at extremely low temperatures. We will see that, as the temperature increases, γ acquires a logarithmic correction $\delta\gamma_{\text{FL}}(T) \propto \ln(1/T)$. Such a correction is in fact observed in a number of heavy-fermion metals at temperatures exceeding a rather low temperature T_0 . Its presence is usually explained within the framework of the collective scenario for the QCP. As already discussed, in this scenario quasiparticle weight z vanishes at a second-order transition point. Consequently, the mass operator $\Sigma(p, \epsilon)$ contains a marginal term $\sim \epsilon \ln \epsilon$, giving rise to the ap-

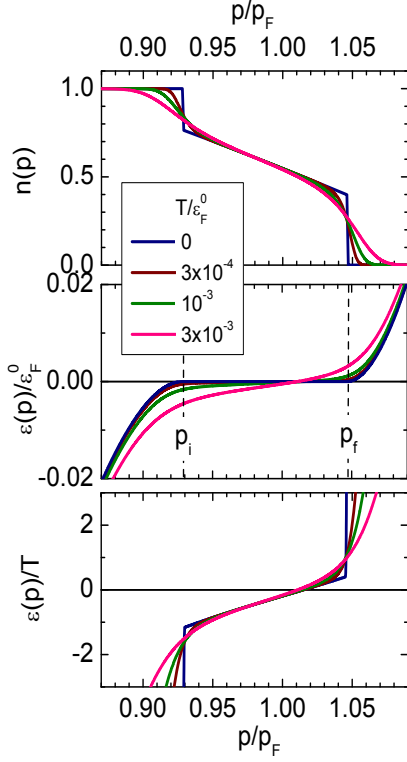


FIG. 14: Single-particle spectrum $\epsilon(p)$ in units of ϵ_F^0 (top panels), occupation number $n(p)$ (middle panels), and ratio $\epsilon(p)/T$ (bottom panels) for the model (72) with the parameters $\beta_* = 10$ and $g_* = 70.0$. All three quantities are shown as functions of p/p_F at different line-type-coded temperatures in units of ϵ_F^0 .

pearance of a logarithmic correction $\delta\gamma_{\text{FL}}(T) \propto \ln(1/T)$ to the FL Sommerfeld ratio $\gamma_{\text{FL}}(T) = \text{const.}$

However, as we will be readily appreciated, breakdown of the quasiparticle picture is not a necessary condition for the emergence of logarithmic corrections to the Sommerfeld ratio $\gamma(T)$. Such corrections can emerge in the QCP region due to the presence of small spin-fluctuation contributions to the FL interaction function. To see this, let us evaluate $\gamma(T)$ within the spin-fluctuation scenario introduced in Sec. VII.B. One has

$$\gamma(T) \sim T^{-1} \int \epsilon(p) \frac{\partial n(p, T)}{\partial T} dv, \quad (73)$$

where $n(p, T) = (1 + \exp(\epsilon(p, T)/T))^{-1}$ is the usual Landau quasiparticle occupation number. Referring to Eq. (65) we know already that at $\xi^{-1} \ll (p - p_F) \ll p_F$, the group velocity $v(p)$ takes on a logarithmic correction $\sim \ln((p - p_F)/p_F) \sim \ln(M^*\epsilon/p_F)$, implying that

$$\frac{1}{v(\epsilon)} \simeq \frac{1}{v_F} \left(1 - \frac{3g^2}{2\pi v_F} \lambda \ln \left(\frac{M^*\epsilon}{p_F} \right) \right). \quad (74)$$

Since the overwhelming contributions to $\gamma(T)$ and other thermodynamic quantities come from the energy region

where $|\epsilon| \simeq T$, we introduce a new variable $y = \epsilon/T$ to obtain the standard FL formula

$$\gamma(T, \rho) \sim \int y^2 n(y) (1 - n(y)) \frac{1}{v(y)} dy. \quad (75)$$

Upon inserting Eq. (74) into this equation, we infer that the logarithmic correction to $\gamma_{\text{FL}} = \text{const.}$ becomes well-pronounced already at a rather low temperature $T_0 = T_F/(p_F \xi)$, with $T_F = p_F^2/2M^*$, according to

$$\frac{\delta\gamma_{\text{FL}}(T)}{\gamma_{\text{FL}}} \simeq \frac{3g^2 M^*}{\pi p_F} \ln \left(\frac{T_F}{T} \right), \quad T_0 < T < T_F. \quad (76)$$

The T -dependence of this correction coincides with that obtained within the conventional scenario for the QCP. But importantly: in contrast to this scenario, the regime involved is *far from* the critical one where the quasiparticle picture breaks down. Similar logarithmic corrections emerge in the same temperature interval when evaluating the thermal expansion coefficient, magnetic susceptibility, and other thermodynamic quantities. Their presence is irrelevant to violation of the quasiparticle picture.

The value of the parameter $(p_F \xi)$, and hence that of the governing parameter λ , can be extracted from experimental data obtained on the upper and lower boundaries of the interval where the logarithmic behavior (76) is in effect. The analysis of these data yields $10 \leq p_F \xi \leq 10^2$, implying that

$$10^2 < \lambda < 10^4. \quad (77)$$

Comparing the Sommerfeld ratios on opposite edges of the interval of logarithmic behavior, one can also estimate the value of another parameter, namely $g^2/(\pi v_F^0)$. Since γ drops by factor 2 or so, we conclude that $g^2/(\pi v_F^0) < 1/\ln(p_F \xi) < 0.1$.

These formulas, in first turn, Eq. (75), are valid provided the inequality $T_0 < T_m$ is met. Otherwise, at $T > T_m$ the FC forms, leading to a dramatic change of the Sommerfeld ratio $\gamma(T)$. Indeed, according to the foregoing analysis, the major part of the FC momentum distribution is T -independent and therefore does not contribute to $\gamma(T)$. Consequently, when formation of the FC is completed, the enhancement factor in $\gamma(T)$ disappears in spite the huge FC density of states.

B. Entropy excess

A drastic change in the behavior of the entropy $S(T)$ occurs^{1,16,23} when, at $T > T_m$, a FC is formed in the domain \mathcal{C} under rearrangement of the momentum distribution $n(p)$ into the self-consistent FC solution $n_*(p)$. The basic entropy formula (3) of the original quasiparticle formalism remains intact:

$$S_* = -2 \iint [n_*(\mathbf{p}) \ln n_*(\mathbf{p}) + (1 - n_*(\mathbf{p})) \ln (1 - n_*(\mathbf{p}))] dv. \quad (78)$$

However, due to the NFL component in $n_*(p)$, the system is seen to possess a T -independent entropy excess $S_*(\rho)$. The situation we now face—with the strongly correlated fermion system having a finite value S_* of the entropy at $T \rightarrow 0$ —resembles that encountered in a system of localized spins. In the spin system, the entropy referred to one spin is simply $\ln 2$, while in the system having a FC, we have $S_*/N \simeq \eta \ln 2$, where $\eta = (p_f - p_i)/p_F$ is the dimensionless FC parameter.

On the other hand, numerical calculations demonstrate that within the FC domain, the momentum distribution $n_*(p)$ changes rapidly under variation of the total density ρ . The corresponding nonzero value of the derivative $\partial S_*/\partial \rho$ produces a huge enhancement of the thermal expansion coefficient $\beta \sim \partial S(T)/\partial \rho \simeq \eta$ with respect to its FL value, proportional to T .⁷⁸ Consonant with this result, in many heavy-fermion metals it is found that β is indeed almost temperature-independent and that it exceeds typical values for ordinary metals by a factor 10^3 – 10^4 .⁷⁹

C. Curie-Weiss behavior of the spin susceptibility

Another peculiar feature of strongly correlated Fermi systems in the QCP region involves the temperature dependence of the spin susceptibility $\chi(T) = \chi_0(T)/(1 + g_0\Pi_0(T))$, where

$$\begin{aligned} \chi_0 &= \mu_e^2 \Pi_0(T) = -2\mu_e^2 \int \frac{dn(p, T)}{d\epsilon(p)} dv \\ &= 2\frac{\mu_e^2}{T} \int n(p, T)(1 - n(p, T)) dv \end{aligned} \quad (79)$$

and g_0 is the spin-spin component of the interaction function. As mentioned before, in the QCP region, the standard FL structure is preserved only at $T < T_m$. In particular, the standard Pauli behavior $\chi(T) = \text{const.}$ is maintained until “melting” of the multi-sheet structure occurs at $T \simeq T_m$, giving rise to flattening of the single-particle spectrum associated with a FC.

At $T > T_m$, insertion of $n_*(p)$ into Eq. (79) yields the Curie-like term

$$\chi_*(T) = \mu_e^2 \frac{C_{\text{eff}}(\rho)}{T} \quad (80)$$

with an effective Curie constant

$$C_{\text{eff}}(\rho) = 2 \int n_*(p)(1 - n_*(p)) dv \quad (81)$$

that depends dramatically on the density.^{23,80} Since C_{eff} is proportional to the FC parameter η , we infer that

$$C_{\text{eff}} \simeq S_*. \quad (82)$$

Thus, all compounds in which the spin susceptibility exhibits the Curie-like behavior possess a large entropy.

Furthermore, in the whole temperature interval from $T = 0$ to $T > T_m$, the spin susceptibility of a Fermi system beyond the QCP possesses Curie-Weiss-like behavior $\chi(T) \sim 1/(T - T_W)$ with a *negative* Weiss temperature T_W . Measurements in ^3He films on various substrates and in numerous heavy-fermion compounds provide examples of this NFL behavior. We emphasize that in our scenario, the negative sign of T_W holds even if the spin-spin interaction is attractive. This contrasts with the behavior in a system of localized spins, where the Weiss temperature has a negative sign only if the spin-spin interaction is *repulsive*. Moreover, in the case of localized spins, the Stoner factor must be suppressed, whereas in the vicinity of the QCP, this factor is usually enhanced.

Another conspicuous feature of the physics beyond the QCP is associated with the Sommerfeld-Wilson ratio $R_{\text{SW}} = T\chi(T)/\mu_e^2 C(T) \sim S_*/C(T)$. Since the excess entropy S_* does not depend on T , it makes no contribution to the specific heat $C(T)$; consequently in systems with a FC, one should see a great enhancement of R_{SW} .

IX. CLASSICAL BEHAVIOR NEAR THE QCP

In previous sections, we have discussed the structure of the extended quasiparticle picture near the QCP and beyond it. In this section, we highlight one of the most distinctive (and counterintuitive) features of strongly correlated Fermi systems in this domain of the Lifshitz phase diagram, notably their classical behavior. This aspect was first revealed in measurements of the specific heat $C(T)$ of dense ^3He films at the lowest temperatures $T \simeq 1$ mK reached experimentally. As the temperature is lowered into this regime, the specific heat $C(T)$ behaves as^{11,12}

$$C(T) = \beta + \gamma T, \quad (83)$$

thus exhibiting an unexpected T -independent term $\beta(\rho)$. Such behavior contrasts sharply with that of its 3D counterpart, bulk liquid ^3He , for which $C(T)$ obeys standard FL theory in approaching zero linearly in T , with a prefactor γ proportional to the density of states $N(0) \propto M^*$. Indeed, FL theory works better and better as T is lowered toward zero, provided no superfluid transition intervenes.

It has been commonly assumed⁸¹ that the term $\beta(\rho)$ is related to the nature of the substrate which supports the 2D ^3He film. More specifically, it is thought that due to weak heterogeneity of the substrate (steps and edges on its surface), quasiparticles are delocalized from it, giving rise to the β feature of the heat capacity.¹² This explanation is undermined somewhat by the fact that if substrate-induced disorder were responsible for the β feature, one would expect the departure of $C(T)$ from the FL prediction to decrease as the film density increases, since the effects of disorder are most pronounced in weakly-interacting systems. Contrariwise, the specific-heat anomaly makes its appearance in just that density region where the effective mass M^* is enhanced and

2D liquid ^3He system becomes strongly correlated.^{11,12} In this situation, the impact of disorder should be suppressed. The weight of these considerations compels us to treat the observed behavior of $C(T)$ as an *intrinsic* property of 2D liquid ^3He .

Curiously, due to the presence of the residual term β , this behavior mimics the classical Dulong-Petit (DP) Law of the specific heats $C(T)$ of solids, the major contribution to which comes from phonons, being obtained from the formula

$$C_B(T) \propto \int_0^{k_{\max}} \omega(k) \frac{\partial n_B(k)}{\partial T} \frac{k dk}{2\pi}, \quad (84)$$

where $n_B(k) = [e^{\omega(k)/T} - 1]^{-1}$ is the Bose-Einstein phonon distribution function. The T -dependence of $C_B(T)$ is determined by relation between the temperature T and the Debye temperature $\Omega_D = \omega(k_{\max}) \simeq ck_{\max}$. At $T \geq \Omega_D$, the specific heat becomes independent of T , and the DP behavior is recovered. The greater the value of the parameter k_{\max} characterizing the cut-off of the phonon spectrum, the more pronounced is the classical contribution β to the specific heat $C(T)$, which is proportional to k_{\max}^D for a system of dimensionality D .

Normally, Ω_D is sufficiently high that, true to its empirical discovery, the DP Law belongs to the realm of classical physics. However, we will see that in strongly correlated Fermi systems, the characteristic frequency of some collective mode can be extremely small such as to allow DP behavior (83) in 2D liquid ^3He at millikelvin temperatures. Certainly, neither the usual hydrodynamic sound nor longitudinal zero sound can qualify, since the group velocities of both modes remain finite as $M^* \rightarrow \infty$.^{82,83} The spin fluctuation mode is excluded as well, however, due to substantial Landau damping. The desired mode is in fact provided by transverse zero sound mode (TZSM),⁸⁴ whose properties near the QCP can be explicated in terms of its well-known dispersion relation in 3D matter⁸²

$$\frac{s}{2} \ln \frac{s+1}{s-1} = \frac{F_1 - 6}{3F_1(s^2 - 1)}, \quad (85)$$

conveniently rewritten as

$$(s^2 - 1) \left(\frac{s}{2} \ln \frac{s+1}{s-1} - 1 \right) = \frac{1}{3} - \frac{2}{F_1}, \quad (86)$$

with $s = c/v_F$ and $F_1 = f_1 p_F M^* / \pi^2$. The TZSM is seen to propagate only if $F_1 > 6$, i.e., provided $M^* > 3M$. Near the QCP where $M^*(\rho) \rightarrow \infty$, this requirement is always met. In this case, one has $v_F/c \rightarrow 0$, and Eq. (85) simplifies to

$$1 = \frac{F_1}{15} \frac{v_F^2}{c^2}, \quad (87)$$

which implies

$$c(\rho \rightarrow \rho_\infty) \rightarrow \sqrt{\frac{p_F v_F(\rho)}{M}} \propto \frac{p_F}{M} \sqrt{\frac{M}{M^*(\rho)}} \rightarrow 0, \quad (88)$$

an analogous formula being obtained for a 2D system.

It should now be clear that toward the QCP, the effective Debye temperature $\Omega_t = \omega(k_{\max}) = c_R k_{\max}$ goes down to zero, independently of the value of the wave number k_{\max} corresponding to the saturation of the TZSM spectrum. Thus, the necessary condition $\Omega_t < T$ for emergence of a regime of classical behavior is always met. However, another condition must also be satisfied if there is to exist a well-pronounced classical domain at extremely low temperature: the parameter k_{\max} must not be too small.

There is no such a restriction in the conventional electron-phonon problem. Indeed, the phonon group velocity c depends weakly on the wave number k , whose characteristic value k_{\max} coincides with its maximum possible value $\simeq 1/r_0$, the inverse distance between particles in the system. As a result, the phonon contribution to the specific heat $C(T)$ turns out to be proportional to the particle density ρ . However, softening of the TZSM is terminated around some rather small critical wave number k_{\max} , so that at greater wave numbers, $\omega(k)$ becomes larger than T , destroying classical behavior. Evaluation of the corresponding threshold value is a difficult problem, because its solution requires analysis beyond the long-wave approximation. Here we consider the situation in systems with only a small proportion of FC, where this problem can be handled with some facility based on the familiar FL kinetic equation^{39,83}

$$(\omega - \mathbf{k}\mathbf{v}) \delta n(\mathbf{p}) = -\mathbf{k}\mathbf{n} \frac{\partial n(\mathbf{p})}{\partial p} \int \mathcal{F}(\mathbf{p}, \mathbf{p}_1) \delta n(\mathbf{p}_1) dv_1. \quad (89)$$

Focusing on transverse zero sound in 2D liquid ^3He , we need only to include the term in the Landau amplitude \mathcal{F} involving the first harmonic f_1 . Making the usual identification $(c_t - \cos \theta) \delta n(\mathbf{p}) = (\partial n(p)/\partial p) \phi(\mathbf{n})$, where $\cos \theta = \mathbf{k}\mathbf{v}/kv$, Eq. (89) becomes

$$\phi(\theta) = -f_1 p_F \cos \theta \int \cos \chi \frac{\partial n(p_1)/\partial p_1}{c_t - v(T) \cos \theta_1} \phi(\theta_1) \frac{dp_1 d\theta_1}{(2\pi)^2}, \quad (90)$$

where $\cos \chi = \cos \theta \cos \theta_1 + \sin \theta \sin \theta_1$, while $v(T)$ is the FC group velocity, proportional to T . The solution describing transverse zero sound is $\phi(\mathbf{n}) \sim \sin \theta \cos \theta$.

It is seen immediately that $c_t \gg v(T) \sim T$; therefore the transverse sound in question does not suffer Landau damping. Then, upon retaining only the leading relevant term $v(T) \cos \theta / c_t^2$ of the expansion of $1/(c_t - v(T) \cos \theta)$ and performing straightforward manipulations, we are led to the simple result

$$c_t^2 \propto -\frac{p_F}{M} \int \frac{\partial n(p)}{\partial p} v(p, T) dp. \quad (91)$$

Factoring out an average value of the group velocity $v(p, T) \simeq T/p_F$, we find that

$$c_t \propto \sqrt{\frac{T}{M}}, \quad (92)$$

i.e., in the FC domain of the phase diagram, the velocity c_t of the transverse mode *depends on temperature* so as to vanish like \sqrt{T} as $T \rightarrow 0$.

We now demonstrate that such a softening of the TZSM holds only as long as the wave number k does not exceed $k_{\max} \simeq L = p_f - p_i$. The reason for this termination is that the *noncondensed* component of the quasiparticle system in 2D liquid ^3He comes into play at $k > L$, consisting of quasiparticles with normal T -independent dispersion $d\epsilon(p)/dp \simeq p_F/M$. The group velocity $c_t(k > L)$ of the transverse mode then soars upward, rendering the corresponding contribution to $C(T)$ irrelevant.

To evaluate this effect it is necessary to go beyond the long-wave approximation, a process beset with replacement of $\mathbf{k}n\partial n(p)/\partial p$ by $n(\mathbf{p} + \mathbf{k}) - n(\mathbf{p})$ and $\mathbf{k}\mathbf{v}$ by $\epsilon(\mathbf{p} + \mathbf{k}) - \epsilon(\mathbf{p})$. With these replacements, simple algebra leads to the behavior

$$\omega^2(k) \propto - \int (n(\mathbf{p}) - n(\mathbf{p} + \mathbf{k})) (\epsilon(\mathbf{p}) - \epsilon(\mathbf{p} + \mathbf{k})) dv. \quad (93)$$

At $k \leq L$, almost all the FC states contribute to this expression on an equal footing, yielding relation (92). However, at $k > L$, the predominant contributions to the integral 93 come from momentum regions where the difference $|\epsilon(\mathbf{p}) - \epsilon(\mathbf{p} + \mathbf{k})|$ has its maximum value, which is T -independent. Thus we infer that at such large k values, softening of the spectrum $\omega_t(k)$ is terminated—in contrast to our previous claim⁸⁴ that softening of $\omega_t(k)$ persists until $k > \sqrt{p_F L}$.

Since, as we have seen, $k_{\max} \simeq L$. Hence we arrive at the result

$$\Omega_t \simeq k_{\max} c_t \propto L \sqrt{\frac{T}{M}}. \quad (94)$$

As long as the inequality $L < (MT)^{1/2}$ holds (or equivalently, $T/\epsilon_F^0 > L/p_F$), the ratio Ω_t/T remains small, and the Dulong-Petit law $C(T) = \text{const.}$ obtains. Thus, in spite of the low temperature, the specific heat behaves as if the system were situated in the classical regime. This paradoxical outcome is a consequence of the presence, in the strongly correlated Fermi system, of a macroscopic subsystem with heavy quasiparticles. As the temperature ultimately goes down to zero at a fixed density ρ , the inequality $L < (MT)^{1/2}$ eventually fails, the quantum regime is restored, and the dominant contribution to the specific heat comes from the “normal” fermions. In other words, by reducing T sufficiently, one can reach a domain in which the FL behavior of the specific heat $C(T)$ is recovered.

Interestingly, the value of the constant term in $C(T)$ can be evaluated in closed form in terms of the FC range L . Upon inserting $\omega_t(k) = c_t k$ into Eq. (84) and integrating, the T -independent term in the specific heat is found to be

$$\frac{C}{N} = \frac{L^2}{8\pi\rho}, \quad (95)$$

where N is the number of atoms in the film.

We turn finally to a discussion of the impact of the TZSM on transport properties. In the case of a conventional Fermi liquid, the Fermi surface consists of a single sheet, so the TZSM has a single branch with velocity c_t exceeding the Fermi velocity v_F . Consequently, emission and absorption of sound quanta by electrons is prohibited, and the role of the TZSM in kinetics is of little interest. However, in heavy-fermion metals, it is common for several bands to cross the Fermi surface simultaneously, thereby generating several zero-sound branches. For all branches but one the sound velocities are less than the largest Fermi velocity. Hence the aforementioned ban is lifted, and these branches of the TZSM spectrum do experience damping, in a situation similar to that for zero-spin sound. In the latter instance, Landau damping is so strong that the mode cannot propagate through the liquid.^{82,83} It will be seen that this is *not* the case for damping of the TZSM, because of the softening of this mode close to the QCP. Due to the softening effect, the contribution of the damped TZSM to the collision integral has the same form as the electron-phonon interaction at room temperature.

To facilitate analysis of damping of the TZSM in systems having a *multi-connected* Fermi surface, we restrict consideration to the case of two electron bands. The TPT is assumed to occur at one of the bands, so that its Fermi velocity, denoted again by v_F , tends to zero, while the Fermi velocity v_o of the other band remains unchanged through the critical density region. The model dispersion relation for the complex sound velocity $c = c_R + ic_I$ becomes

$$1 = \frac{F_1}{6} \left[1 - 3 \left(\frac{c^2}{v_F^2} - 1 \right) \left(\frac{c}{2v_F} \ln \frac{c + v_F}{c - v_F} - 1 \right) \right] + \frac{F_1}{6} \frac{v_F}{v_o} \left[1 - 3 \left(\frac{c^2}{v_o^2} - 1 \right) \left(\frac{c}{2v_o} \ln \frac{c + v_o}{c - v_o} - 1 \right) \right]. \quad (96)$$

It can easily be verified that the contribution of the second term to the real part of the right-hand side of Eq. (96) is small compared to that of the first term, since $v_F/v_o \rightarrow 0$ toward the QCP. On the other hand, noting that $\ln[(c_R + ic_I + v_o)/(c_R + ic_I - v_o)] \simeq -i\pi$, the corresponding contribution $i\pi F_1 v_F c_R / (4v_o^2)$ to the imaginary part of the right-hand side cannot be ignored, else $c_I = 0$. By this reasoning, Eq. (96) assumes the simplified form

$$1 = \frac{F_1}{15} \frac{v_F^2}{(c_R + ic_I)^2} - i \frac{\pi}{4v_o^2} F_1 v_F c_R \quad (97)$$

analogous to Eq. (87). Its solution obeys

$$c_R \propto \sqrt{\frac{M}{M^*(\rho)}}, \quad c_I \propto \frac{M}{M^*(\rho)}. \quad (98)$$

Importantly, we see then that the ratio $c_I/c_R \propto \sqrt{M/M^*(\rho)}$ is *suppressed* in the QCP regime, which allows us to analyze the contribution of the TZSM to the

collision term entering the resistivity along the same lines as in the familiar case of the electron-phonon interaction.

X. CONCLUSION

Proceeding from the original FL quasiparticle picture due to Landau and Migdal, we have addressed the formation of flat bands in strongly correlated Fermi systems beyond a point where the necessary condition for stability of the Landau state is violated and hence subject to rearrangement. Responding to a soaring exhortation from Migdal, “*Beri shire!*” (Embrace everything you can!), expressed to an interlocutor when A. B. approved what he heard, we have analyzed this phenomenon in diverse strongly correlated Fermi systems from neutron stars to atomic liquids to electron systems of solids. Absent, however, are reviews of the latest achievements in the investigation of flattening of single-particle spectra in equations of particle physics,⁶² in topological media,^{67,85,86} including the analysis of flat bands on the surface of multilayered graphene (see Refs. 87,88 and references therein).

We have seen that in the class of systems under consideration, rearrangement of the Landau state occurs by means of a cascade of topological transitions, at which the number of sheets of the Fermi surface grows steadily, as the spectrum of single-particle excitations $\epsilon(p)$ acquires additional zeroes, in effect becoming flatter and flatter. Tracing the evolution of this spectrum as the coupling constant increases, we have shown that at zero temperature, the salient feature of the final stage of the evolution is the formation of the flat bands whose spectrum $\epsilon(p)$ proves to be dispersionless at $T = 0$, while at low $T \neq 0$ its dispersion becomes proportional to T .

As we have seen in Sec. III, the underlying reason for such a universal rearrangement is based on the Pitaevskii identity,^{40,41} derived from the Galilean invariance of the Hamiltonian of the system and gauge invariance, an identity which coincides with the Landau equation for the spectrum $\epsilon(p)$ as deduced from the assumption that $\epsilon(p)$ is a functional of the quasiparticle momentum distribution $n(p)$. This coincidence establishes that the ground-state energy E is indeed a functional of n , because, according to the Lehmann expansion, $\epsilon(p, n)$ is a variational derivative of E with respect to n . If one does not care about obedience to the Pauli principle, the global minimum of the functional $E(n)$ is then attained at some continuous function $n_*(p)$, devoid (as a rule) of jumps in momentum space. But we know that in weakly or moderately correlated Fermi systems, such solutions violate the Pauli restriction $n(p) \leq 1$, and, as per standard FL theory, the true quasiparticle momentum distribution turns out to be the step function $\theta(p_F - p)$.

In strongly correlated Fermi systems, as we have seen in Sec. V.A, the situation changes drastically: beginning with a critical coupling constant λ_{FC} , the inequality $0 \leq n(p) \leq 1$ is met, lifting the theoretical ban on the emergence of such an exotic creation as the fermion con-

densate.

Establishing this freedom does not in itself qualify as a rigorous proof of the viability of the swelling scenario, as such a proof must surmount obstacles associated with the interplay between single-particle and collective degrees of freedom. The flattening of the single-particle spectrum may lead to softening of some spin/density-fluctuation mode, in principle generating a corresponding second-order phase transition before fermion condensation sets in. Alas, the detailed circumstances of this interplay await proper clarification. Thus it seems that we are forced to seek guidance from analysis of the available experimental data, even though, according to Migdal’s criterion, these data are generally of secondary importance.

The flattening phenomena observed experimentally are so pervasive, however, that an exception to A.B.’s policy statement is in order. The discovery of the quantum critical point, made almost simultaneously in 2D liquid ^3He , MOSFETs, and heavy-fermion metals,^{8,14,15,47} has been a milestone in experimental exploration of strongly correlated Fermi systems, providing a vigorous impetus to their theoretical investigation. Pivotal experimental guidance to theoretical development has emerged from evidence for the separation of QCPs from points of putative second-order phase transitions, uncovered first in 2D liquid ^3He ¹² and more recently in heavy-fermion metals.⁵⁶ Thus separation proves that collective degrees of freedom are, after all, not of crucial importance to the QCP phenomena under study.

At this juncture in the development of QCP physics, one is prompted to ask whether we are facing a situation similar to that surrounding the discovery of the W boson if in fact the phenomenon of flattening of single-particle spectra has already been observed in experiments on condensed-matter systems. If so, then the misinterpretation of these experiments has impeded the revelation of a fundamentally new class of Fermi liquids. Unfortunately, the scope of the data available for sufficiently definitive analysis is still quite limited. Measurements of angle-resolved photoemission electron spectra (ARPES) in solids, while otherwise promising, are not yet accurate enough to confirm or refute the linearity in T of the low-temperature dispersion of these spectra in relevant cases.

Valuable insights might also be drawn from measurements of magnetic oscillations. In these measurements, the electron effective mass is extracted by performing a Lifshitz-Kosevich fit to the temperature dependence of the magnitude of observed oscillations. In strongly correlated electron systems of high- T_c superconductors and heavy-fermion metals possessing a QCP, such analysis is as yet limited to isolated examples. To our knowledge, a QCP has so far been documented only in the single high- T_c superconductor $\text{YbBa}_2\text{Cu}_3\text{O}_{6+x}$, at a critical doping $x_c \simeq 0.5$ ⁸⁹ where the measurements have been carried out at a temperature around 1 K in strong magnetic fields $H > 55$ T. Additionally, measurements⁹⁰

performed at much lower temperatures around 20 mK on the heavy-fermion metal CeCoIn₅, which also has a QCP, have revealed deviations from the Lifshitz-Kosevich formula itself. Unfortunately, interpretation of the data on magnetic oscillations is burdened due to the necessity of imposing strong magnetic fields, which causes substantial distortion of the original electron motion, triggering magnetic breakdown.⁹¹

In closing, let us briefly revisit the theoretical challenge presented by the growing body of thermodynamic measurements on strongly correlated Fermi systems. We focus on the following striking features of the inferred thermodynamic behavior at temperatures exceeding what may be interpreted as the critical temperature T_m for melting of the FL structure and formation of a flat segment of the quasiparticle spectrum $\epsilon(p)$:

- (i) The existence of a T -independent entropy excess S_* , reflected for example in a huge enhancement upon the FL value of the thermal expansion coefficient $\beta \simeq \beta_* \propto \partial S_*/\partial P$.
- (ii) A sharp falloff of the specific heat $C(T)$ upon passing through the critical temperature T_m .
- (iii) Curie behavior of the magnetic susceptibility $\chi(T) = C_{\text{eff}}/T$, with a effective Curie constant differing from the conventional value.

We now comment specifically on each of these behaviors in turn, within the context of the extension of the Landau-Migdal quasiparticle theory to embrace topological phase transitions, and especially fermion condensation.

- (i) In ordinary FL theory, the entropy S is given by Eq. (3); the curve $S(T)$ starts at the origin and rises linearly with T . To attain entropy values $S \simeq \ln 2$ at low T , the density of states, which specifies the slope $dS(T)/dT$, must be further enhanced beyond the standard heavy-fermion boost associated with the width of the narrow f -band lying exactly at the Fermi surface. Nevertheless, experimental measurements⁹² indicate that the entropy value $S = 0.5 \ln 2$ is realized in the heavy-fermion metal YbIr₂Si₂ at $T > T_m \simeq 1$ K. Moreover, another apparent manifestation of the flattening phenomena is exhibited by the heavy-fermion metal CeCoIn₅ mentioned above. Its thermal expansion coefficient $\beta(T)$, measured in external magnetic fields $H \simeq 5T$ imposed to suppress superconductivity, increases strongly until T reaches the value $T_m = 0.3$ K,⁹³ whereas $T > T_m$, $\beta(T)$ becomes T -independent, being enhanced by a factor $10^3 - 10^4$ compared with standard FL values.
- (ii) A drop of the Sommerfeld ratio $\gamma(T)$ is observed in many strongly correlated electron systems as a temperature T_m is exceeded. The record slump is found in YbIr₂Si₂ where the FL contribution to

$\gamma(T)$ collapses at $T > T_m = 0.7$ K. Usually such a slump is attributed to some second-order (antiferromagnetic) phase transition. (As is well known, in the Landau theory of second-order phase transitions there is a jump of the specific heat at the transition point.) Significantly, the experimental data on this compound and any other compounds having the QCP fail to show any jump of $C(T)$ at any T . The scaling theory of second-order phase transitions also fails to explain the experimental behavior of $C(T)$. In addition, numerous attempts to establish the structure of the corresponding order parameter on the side of T_m allegedly associated with the ordered phase, have also been unsuccessful. In light of all that has been done before in this article, none of this comes as a big surprise, and reminds us of a saying by Confucius:

It is hard to find a black cat in a dark room, especially if it is not there.

Indeed, the arguments given in Sec. V.E explain the behavior of $C(T)$ in the vicinity of T_m , attributing it to a *crossover* from a state with a multi-connected Fermi surface to a state having a *flat band*. Bearing in mind this association, we infer that in the QCP region, there are no hidden order parameters at all, since the behavior stems from topological phase transitions rather than second-order, symmetry-breaking phase transitions.

- (iii) Curie-like behavior of the magnetic susceptibility $\chi(T > T_m) = C_{\text{eff}}/T$, which is incompatible with the customary FL Pauli behavior $\chi(T) = \text{const}$, was first observed in 2D liquid ³He.¹⁰ An analogous Curie-like behavior of $\chi(T)$ has also been seen recently at very low temperatures in normal states of high- T_c superconductors placed into a strong magnetic field to suppress the superconductivity.^{94,95} In both the cases, the effective Curie constant C_{eff} , being nontrivially dependent on the density ρ or doping x , is at variance with the ordinary Curie constant proportional to ρ .

A detailed discussion of experimental support for or against the topological scenario for the quantum critical point is beyond the scope of this article, especially bearing in mind that the measurements involved are for the most part very fresh, and—as cautioned by A. B. Migdal—may be subject to error or incomplete analysis. Thus, in summarizing the current state of knowledge, we are forced to recognize that the envisioned swelling of the Fermi surface in most strongly correlated condensed-matter many-fermion systems featuring the occurrence of the flat bands, has been neither validated nor disproved experimentally. The fermion condensate remains as elusive as the Cheshire Cat of *Alice in Wonderland*, teasing us with mischievous grins that become more and more visible and numerous.

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